The total rank question in knot Floer homology and some related observations

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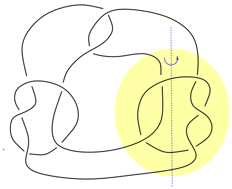
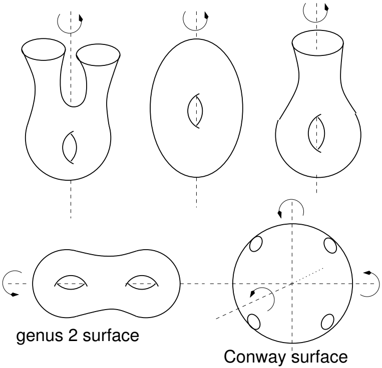
Mutation of M along (F, τ)

Conway mutation of
$K \subset S^3$



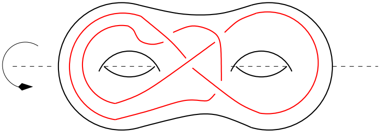
The Kinoshita-Terasaka knot $KT_{2,1}$ as it appears in [6]



-Ruberman [8]

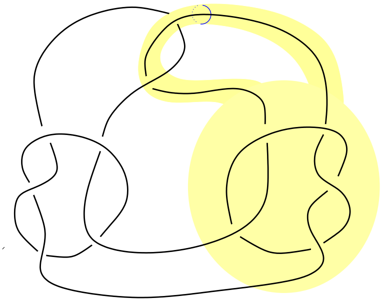
Genus two mutation of a manifold:

- Cut M along F.
- Involute F by τ.
- Glue in $\tau(F)$, get M^{τ}.



Genus two mutation of a knot:

- $F = \partial H, H \supset K$.
- Mutate S^3, get S^3 back.
- Mutant knot is K^{τ}.



Heegaard Floer and Khovanov knot homologies for $K \subset S^3$

$$K \rightsquigarrow \widehat{\mathrm{CFK}}(K) \rightsquigarrow \widehat{\mathrm{HFK}}$$

$$K \rightsquigarrow \mathrm{CKh}(K) \rightsquigarrow \mathrm{Kh}$$

[Ozsváth and Szabó, Rasmussen, Khovanov]

Conway mutation as a genus two mutation.

Theorem (Ozsváth and Szabó [6])

$$\widehat{\mathrm{HFK}}(KT) \ncong \widehat{\mathrm{HFK}}(C)$$

$\widehat{\mathrm{HFK}}(KT)$

	−2	−1	0	1	2
3					$\mathbb{F}$
2				$\mathbb{F}^4$	$\mathbb{F}$
1			$\mathbb{F}^6$	$\mathbb{F}^4$	
0		$\mathbb{F}^4$	$\mathbb{F}^7$		
−1	$\mathbb{F}$	$\mathbb{F}^4$			
−2	$\mathbb{F}$				
dim = 33					

$\widehat{\mathrm{HFK}}(C)$

	−3	−2	−1	0	1	2	3
4							$\mathbb{F}$
3						$\mathbb{F}^3$	$\mathbb{F}$
2					$\mathbb{F}^3$	$\mathbb{F}^3$	
1				$\mathbb{F}^2$	$\mathbb{F}^3$		
0			$\mathbb{F}^3$	$\mathbb{F}^3$			
−1		$\mathbb{F}^3$	$\mathbb{F}^3$				
−2	$\mathbb{F}$	$\mathbb{F}^3$					
−3	$\mathbb{F}$						
dim = 33							

Observation

$$\bigoplus \dim_{m,s} \widehat{\mathrm{HFK}}(K) = \bigoplus \dim_{m,s} \widehat{\mathrm{HFK}}(K^{\tau})$$

Theorem (Ozsváth and Szabó [6])

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-1	$\mathbb{F}$	$\mathbb{F}^4$			
-2	$\mathbb{F}$				
$\dim = 33$					

$\widehat{HFK}(C)$							
	-3	-2	-1	0	1	2	3
4							$\mathbb{F}$
3						$\mathbb{F}^3$	$\mathbb{F}$
2					$\mathbb{F}^3$	$\mathbb{F}^3$	
1				$\mathbb{F}^2$	$\mathbb{F}^3$		
0			$\mathbb{F}^3$	$\mathbb{F}^3$			
-1		$\mathbb{F}^3$	$\mathbb{F}^3$				
-2	$\mathbb{F}$	$\mathbb{F}^3$					
-3	$\mathbb{F}$						
$\dim = 33$							

Observation

$$\bigoplus \dim_{m,s} \widehat{HFK}(K) = \bigoplus \dim_{m,s} \widehat{HFK}(K^\tau)$$

The question of total rank invariance

Is the total rank of knot Floer homology or Khovanov homology invariant under mutation?

Theorem with Starkston

Theorem (M. and Starkston)

There exist infinitely many knots K_n admitting a nontrivial genus two mutant K_n^τ such that:

- $\widehat{\mathrm{HFK}}_m(K_n, s) \cong \widehat{\mathrm{HFK}}_m(K_0, s)$ *and* $\widehat{\mathrm{HFK}}_m(K_n^\tau, s) \cong \widehat{\mathrm{HFK}}_m(K_0^\tau, s)$ *for all* n *.*
- K_n *and* K_n^τ *are distinguished by both* $\widehat{\mathrm{HFK}}$ *and* Kh *as a bigraded groups.*
- K_n *and* K_n^τ *are distinguished by* δ*-graded* $\widehat{\mathrm{HFK}}$ *and* Kh*.*
- *Each genus two mutant pair* (K_n, K_n^τ) *has the same total dimension with respect to each of these invariants.*

What our theorem shows:

$$\widehat{\mathrm{HFK}}(K_n) \xrightarrow[\tau \text{ preserves } \bigoplus \dim]{} \widehat{\mathrm{HFK}}(K_n^\tau)$$

$$\cong \qquad\qquad\qquad \cong$$

$$\widehat{\mathrm{HFK}}(K_0) \xrightarrow[\tau \text{ preserves } \bigoplus \dim]{} \widehat{\mathrm{HFK}}(K_0^\tau)$$

$$\cong \qquad\qquad\qquad \cong$$

$K_0 = 14^n_{22185}$

	-2	-1	0	1	2
3					$\mathbb{F}$
2				$\mathbb{F}^2$	$\mathbb{F}$
1			$\mathbb{F}^2$	$\mathbb{F}^2$	
0		$\mathbb{F}^2$	$\mathbb{F}^3$		
-1	$\mathbb{F}$	$\mathbb{F}^2$			
-2	$\mathbb{F}$				

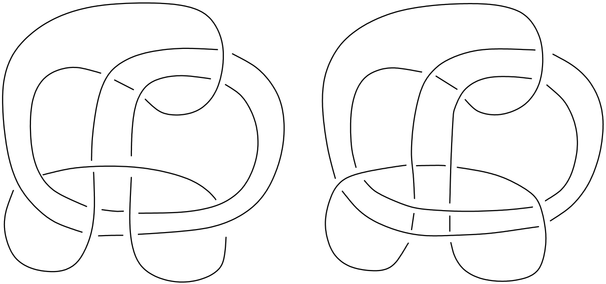
dim $= 17$

$K_0^\tau = 14^n_{22589}$

	-1	0	1
1			$\mathbb{F}^2$
0		$\mathbb{F}^5$	$\mathbb{F}^2$
-1	$\mathbb{F}^2$	$\mathbb{F}^4$	
-2	$\mathbb{F}^2$		

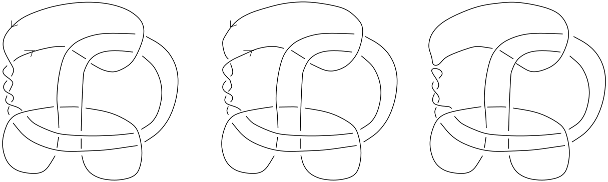
dim $= 17$

Genus two mutant pair



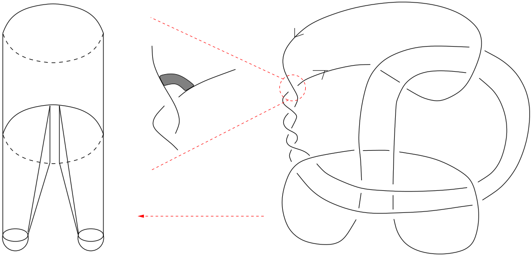
These genus two mutant knots are not Conway mutants.

Special property: Skein triples



Oriented: $(K_n, \quad K_{n-2}, \quad unlink)$
Unoriented: $(K_n, \quad K_{n-1}, \quad unlink)$

Special property: K_n is slice



$$|\tau| \leq g_*(K_n) = 0$$

That's Ozsváth and Szabó's τ, not the mutation τ!

Using an observation of Hedden [4] about the Skein exact sequence of $\widehat{\mathrm{HFK}}$:

$$0 \longrightarrow \mathrm{HFK}^-_{1-2d}(K_n, -d) \xrightarrow{f^-} \mathrm{HFK}^-_{1-2d}(K_{n-2}, -d) \xrightarrow{g^-} \mathbb{F}_{2\{-2d,-d\}} \xrightarrow{h^-} \mathrm{HFK}^-_{-2d}(K_n, -d) \xrightarrow{i^-}$$

$$U^d \cdot z \longmapsto U^d \cdot \xi_n + \eta$$

$$\mathrm{HFK}^-_{-2d}(K_{n-2}, -d) \xrightarrow{j^-} \mathbb{F}_{2\{-1-2d,-d\}} \xrightarrow{k^-} \mathrm{HFK}^-_{-1-2d}(K_n, -d) \xrightarrow{\ell^-} \mathrm{HFK}^-_{-1-2d}(K_{n-2}, -d) \longrightarrow 0$$

$$U^d \cdot \xi_{n-2} \longmapsto U^d \cdot z'$$

The proof is similar in Khovanov homology (Slice, $s = 0$, Skein exact sequence ...).

Applications:

Conjecture (Baldwin and Levine [1])

δ-graded $\widehat{\mathrm{HFK}}$ is invariant under Conway mutation.

- Our theorem shows their conjecture does not extend to genus two mutation.

δ − graded $\widehat{\mathrm{HFK}}(K_0)$						
	−2	−1	0	1	2	dim
$s-m=-1$	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	8
$s-m=0$	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	9
dim = 17						

δ − graded $\widehat{\mathrm{HFK}}(K_0^\tau)$				
	−1	0	1	dim
$s-m=0$	$\mathbb{F}^2$	$\mathbb{F}^5$	$\mathbb{F}^2$	9
$s-m=+1$	$\mathbb{F}^2$	$\mathbb{F}^4$	$\mathbb{F}^2$	8
dim = 17				

What if total rank of $\widehat{\mathrm{HFK}}$ is indeed preserved under Conway mutation?

- Recall that any knot admitting L-space surgeries has restricted $\widehat{\mathrm{HFK}}$ [7]; $\forall s$

$$\widehat{\mathrm{HFK}}(K, s) \cong \mathbb{Z} \text{ or } 0$$

- Mutant knots share the same Alexander polynomial.

$\Rightarrow$ Any mutant of an L-space knot also an L-space knot.

*In general, mutants of fibered knots are not even necessarily fibered!

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This begs the question

Do L-space knots admit nontrivial mutations... at all?

New project!

Goal: Demonstrated that the knot Floer complex "sees" essential Conway spheres.

Conjecture (Lidman and M.)

If $S^3 - K$ contains an essential four-punctured sphere, then there exists an Alexander grading s such that rank $\widehat{\mathrm{HFK}}(K, s) \geq 2$.

Implies:

Conjecture (Lidman and M.)

Suppose that K is a knot in S^3 with an L-space surgery. Then, $S^3 - K$ does not contain an essential four-punctured sphere.

Corollary

L-space knots admit no nontrivial mutations.

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Progress

Claim (Lidman and M.)

The conjecture is true for pretzel knots.

This is not a surprising result:
In [7], $P(-2, 3, n)$ are shown to admit L-space surgeries.

Proof (work in progress)

Gabai's classification of fibered pretzel links [3].

Alexander polynomial obstructions:

- Show $det(K) > 2g + 1$ via Goertiz matrices or $H_1(\Sigma_2(K))$.
- Counting arguments via Kauffman states of the knot diagram.
- Work of Hironaka with Lehmer polynomials [5].

Future work: Address the larger conjecture with bordered Floer homology and sutured techniques.

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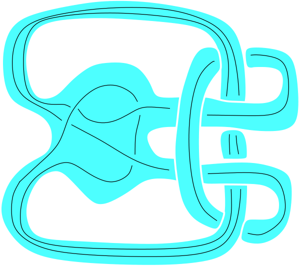

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John A. Baldwin and Adam Simon Levine.
A combinatorial spanning tree model for knot Floer homology.
To appear in Adv. in Math., 2011.
arXiv:1105.5199v2 [math.GT].

Nathan M. Dunfield, Stavros Garoufalidis, Alexander Shumakovitch, and Morwen Thistlethwaite.
Behavior of knot invariants under genus 2 mutation.
New York J. Math., 16:99–123, 2010.

David Gabai.
Detecting fibred links in S^3.
Comment. Math. Helv., 61(4):519–555, 1986.

Matthew Hedden and Liam Watson.
On the geography and botany of knot Floer homology, Preprint.

Eriko Hironaka.
The Lehmer polynomial and pretzel links.
Canad. Math. Bull., 44(4):440–451, 2001.

Peter Ozsváth and Zoltán Szabó.
Knot Floer homology, genus bounds, and mutation.
Topology Appl., 141(1-3):59–85, 2004.

Peter Ozsváth and Zoltán Szabó.
On knot Floer homology and lens space surgeries.
Topology, 44(6):1281–1300, 2005.

Daniel Ruberman.
Mutation and volumes of knots in S^3.
Invent. Math., 90(1):189–215, 1987.