Cosmetic surgery in L-spaces and nugatory crossings

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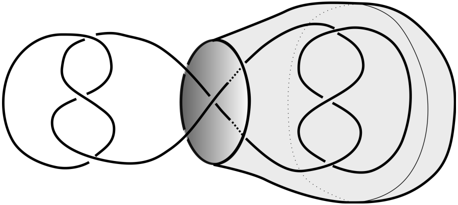
K is a tame, oriented knot in S^3.
Notice: $K^+ \simeq K^-$.



$D =$ is the crossing disk,
alg. $\#(D \cap K) = 0$

$\partial D = C$ is the crossing circle

A *cosmetic* crossing change preserves the isotopy type of K.
A crossing c is *nugatory* iff C bounds embedded disk in $S^3 - K$.

The Cosmetic (aka "Nugatory") Crossing Conjecture

Conjecture (X. S. Lin)

If K admits a crossing change at c which preserves the oriented isotopy class of the knot, then c is nugatory.

Remark

This is Problem 1.58 on the "Kirby List."

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Orientation and mirroring matter. Otherwise, consider $K^+ = P(-3, -1, 3)$. Then $K^+ \simeq -K^-$.

What is known?

Families of knots known to satisfy the CCC:

- The unknot (Scharlemann and Thompson)
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Other obstructions:

- If $g(K) = 1$ and K admits a cosmetic crossing change, K is algebraically slice. (BFKP)
- Winding number zero satellites of prime, non-cables with pattern satisfy CCC also satisfy CCC. For example, Whitehead doubles. (Balm-Kalfagianni)

What about alternating knots?

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Theorem (First Tait Conjecture*)

Reduced alternating diagrams are minimal, and minimal diagrams of prime, alternating knots are alternating.

*proved by Kauffman, Thisthlethwaite, Murasugi in the 80s

Thus if a cosmetic crossing exists, it must appear in a non-alternating diagram.

What can we utilize instead? Khovanov homology

- $\overline{Kh}^{i,j}(L)$ is reduced Khovanov homology of $L \subset S^3$ over $\mathbb{Z}/2$. Delta-graded variant: $\overline{Kh}^{\delta}(L)$, $\delta = j - i$.
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Theorem (Lee, Manolescu-Ozsváth)

Alternating and quasi-alternating knots are thin.

Theorem (Ozsváth and Szabó)

If K is $\overline{Kh}$–thin then $\Sigma(K)$ is an L-space.

$\Sigma(K)$ is a rational homology sphere,

$$H_1(\Sigma(K); \mathbb{Z}) \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k. \tag{1}$$

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Theorem (Lidman-M.)

Let $K \subset S^3$ be a knot with $\Sigma(K)$ an L-space. If each d_i is square-free, then K satisfies the cosmetic crossing conjecture.

Applications of the theorem

1. **Small knots:** Every knot with at most 9 crossings (and most with 10) satisfies the CCC.

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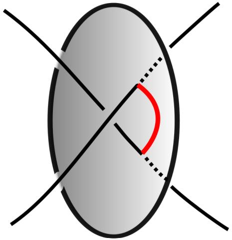
4. **Symmetric unions:** For example, $K_n(5_2)$ when $n \cong 0 \pmod 7$. These have fixed determinant.

Proof Sketch

Suppose K admits cosmetic crossing change at c.



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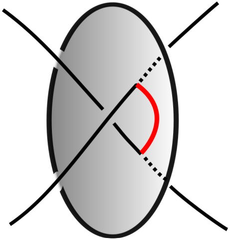

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Suppose $[\tilde{\gamma}] = 0$ in $H_1(\Sigma(K))$. Now we can invoke the Dehn surgery characterization of the unknot

Surgery characterization of the unknot

Let's suppose we know that $[\tilde{\gamma}] = 0$ in $H_1(\Sigma(K))$. We can then apply:

Theorem (Gainullin; Kronheimer-Mrowka-Ozsváth-Szabó)

Let K be a null-homologous knot in an L-space Y. If $Y_{p/q}(K) \cong Y_{p/q}(U)$ then $K \simeq U$.

How is this applied?

Let $M = \Sigma(K) - N(\tilde{\gamma})$, and consider two filling slopes, α and β, where

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Then, the Montesinos trick implies

- $\Delta(\alpha, \beta) = 2$.
- $-1/2$–surgery on $\tilde{\gamma}$: $\Sigma(K^+) \rightsquigarrow \Sigma(K^-)$
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Lemma

$\tilde{\gamma}$ is an unknot in $\Sigma(K)$ implies the crossing c is nugatory.

(This follows as a special case of the $\mathbb{Z}_2$-equivariant Dehn's Lemma of Kim-Tollefson, Gordon-Litherland and Meeks-Yau.)

So why is the lift null-homologous?

Proposition

$[\tilde{\gamma}] = 0$ *in* $H_1(\Sigma(K))$.

Take (μ, λ_M) as a basis for $H_1(\partial M)$, where λ_M is the rational longitude and $\mu \cdot \lambda_M = 1$

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In this basis:

$$\alpha = p\mu + q\lambda_M$$
$$\beta = r\mu + s\lambda_M$$

An observation of Watson's is that λ_M controls the order of the first homology of a Dehn filling. In particular:

$$|H_1(M(\alpha))| = c_M \Delta(\alpha, \lambda_M)$$

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Since $p \neq 2$, we have $p = 1$. Thus

$$|H_1(\Sigma(K))| = c_M = \mathrm{ord}_H\, i_*(\lambda_M) \cdot |H|$$

A special case of our theorem:

As a special case, consider when $\det(K)$ is square-free. (i.e. all of the d_i in decomposition are primes, rather than square-free).

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But $i_*(\lambda_M)$ generates a subgroup of H, thus

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The **general argument** uses some linear algebra to argue that $\mathrm{ord}_H\, i_*(\lambda_M) = 1$ and reaches the same conclusion.

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3. The Cosmetic Crossing Conjecture is still open!

Thank you!