# Montesinos knots, Hopf plumbings and L-space surgeries

Kenneth Baker [‡] Allison Moore[†]

[†]Rice University

[‡]University of Miami

March 29, 2014

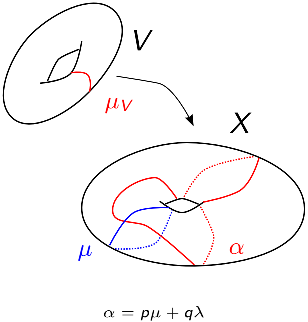
# A longstanding question: Which knots admit lens space surgeries?

$K \subset S^3$, $S^3_{p/q}(K)$ is $(p, q)$-**Dehn surgery** along $K$.

Recall:

$$H_1(S^3_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z} \cong H_1(L(p, q); \mathbb{Z})$$

So $H_1$ doesn't help answer this question at all.



# A bit of history

**Which knots admit lens space surgeries?**

1971 (Moser) torus knots.

1977 (Bailey-Rolfsen) iterated torus knot.

1980 (Fintushel-Stern) hyperbolic knot $(-2, 3, 7)$.

1990 (Berge) many examples.

Cyclic Surgery Theorem (CGLS) + Berge's construction = 'The Berge Conjecture.'

# Knot Floer homology

(Ozsváth-Szabó, Rasmussen):

$$\begin{array}{ccc} K \subset Y & \longrightarrow & \cdots \subset \mathcal{F}_{i-1}C \subset \mathcal{F}_i C \subset \ldots \\ & & \downarrow \\ & & H_*(\mathcal{F}_i C / \mathcal{F}_{i-1} C) \\ & & \| \\ & & \widehat{\mathrm{HFK}}(K) = \bigoplus_{m,s} \widehat{\mathrm{HFK}}_m(S^3, K, s). \end{array}$$

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$$

Facts:

- $\Delta_K(t) = \sum_s \chi(\widehat{\mathrm{HFK}}(K,s)) \cdot t^s$
- $\operatorname{rank} \widehat{HF}(L(p,q)) = p = |H_1(L(p,q))|$

A $\mathbb{Q}HS^3$ $Y$ is an **L-space** if $|H_1(Y;\mathbb{Z})| = \operatorname{rank} \widehat{HF}(Y)$.
Ex: $S^3$, all lens spaces, 3-manifolds with finite $\pi_1$.

# Motivating question recast

**Question**

*Which knots admit lens space surgeries?*

becomes

**Question**

*Which knots admit L-space surgeries?*

# L-space surgery obstructions

**Theorem (Ozsváth-Szabó)**

*If $K$ admits an L-space surgery, then for all $s \in \mathbb{Z}$, $\widehat{HFK}(K, s) \cong \mathbb{F}$ or $0$ (and some other conditions on Maslov grading).*

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**Corollary (Determinant-genus inequality)**

*If $det(K) > 2g(K) + 1$, then $K$ is not an L-space knot.*

# L-space surgery obstructions

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## Corollary (Determinant-genus inequality)

*If $\det(K) > 2g(K) + 1$, then $K$ is not an L-space knot.*

## Proof.

If $K$ is an L-space knot, then $|a_s| \leq 1$ $\forall$ coefficients $a_s$ of $\Delta_K(t)$. Then,

$$\det(K) = |\Delta_K(-1)| \leq \sum_s |a_s| \leq 2g(K) + 1.$$

$\square$

# More obstructions

**Theorem (Ni, Ghiggini)**

*$K$ is fibered if and only if* $\widehat{\mathrm{HFK}}(K, g(K)) \cong \mathbb{F}$.

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Thus $L$-space knots are fibered.

**Theorem (Hedden)**

*An $L$-space knot $K$ supports the tight contact structure; equivalently, an $L$-space knot is strongly quasipositive.*

# Classification theorem

**Theorem (Baker-M.)**

*Among the Montesinos knots, the only L-space knots are the pretzel knots* $P(-2, 3, 2n+1)$ *for* $n \geq 0$ *and the torus knots* $T(2, 2n+1)$ *for* $n \geq 0$.

# Montesinos knots

$$K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2,} \ldots, \frac{\beta_r}{\alpha_r} \,\middle|\, e\right)$$

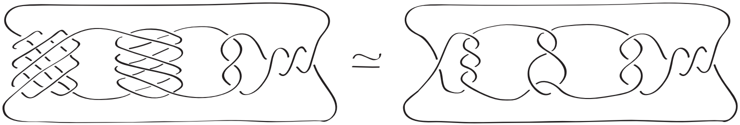

Figure: $M(\frac{3}{4}, -\frac{2}{5}, \frac{1}{3}|3)$.

Where $\alpha_i, \beta_i, e \in \mathbb{Z}$ and $\alpha_i > 1$, $|\beta_i| < \alpha_i$, and $\gcd(\alpha_i, \beta_i) = 1$.

# Ingredients for proof

We need only consider fibered, non-alternating Montesinos knots,

$$K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2,} \ldots, \frac{\beta_r}{\alpha_r} \mid e\right)$$

and we assume $r \geq 3$, because $r \leq 2$ implies $K$ is a two-bridge link.

**Theorem (Ozsváth-Szabó)**

*An alternating knot admits an L-space surgery if and only if $K \simeq T(2, 2n+1)$, some $n \in \mathbb{Z}$.*

# Fibered Montesinos knots

(Hirasawa-Murasugi): Classified fibered Montesinos knots with their fibers. For $K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2,} \ldots, \frac{\beta_r}{\alpha_r} \mid e\right)$,

$$\frac{\beta_i}{\alpha_i} = \cfrac{1}{x_1 - \cfrac{1}{x_2 - \cfrac{1}{\ddots - \cfrac{1}{x_m}}}}$$

$$S_i := [x_1, \ldots, x_m]$$

Two cases:

1. $\alpha_i$ are all odd $\rightsquigarrow$ strict continued fractions.
2. $\alpha_1$ is even, $\alpha_i$ is odd for $i > 1$ $\rightsquigarrow$ even continued fractions.

# Open books for three-manifolds

$(Y, F)$ —an open book for closed 3-manifold $Y$; $L = \partial F$.
$\xi = \ker \alpha$, $\alpha \wedge d\alpha \neq 0$ —a contact structure on $Y$.

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- (Thurston-Winkelnkemper - 1975)
  Every $(Y, F)$ induces a contact structure.
- (Giroux - 2002)
  $\{\xi \text{ on } Y\}/\text{ isotopy} \longleftrightarrow \{(Y, F)\}/\text{ positive stabilization}$

# Plumbings of Hopf bands

Pos/neg (de)stabilization $\leftrightarrow$ (de)plumbing of pos/neg Hopf bands.

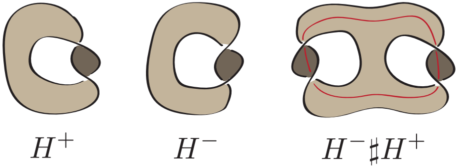


Figure: The connected sum of positive and negative Hopf bands, which admits a Stallings twist along the red curve.

**Lemma (Contact Structures Lemma)**

1. *(Giroux) If $F$ contains a positive Hopf band, then $(Y, F)$ supports the same contact structure as the open book obtained by deplumbing that positive Hopf band.*

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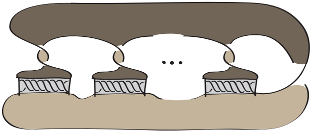
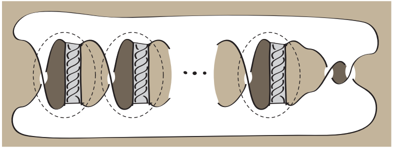
1. *(Giroux) If $F$ contains a positive Hopf band, then $(Y, F)$ supports the same contact structure as the open book obtained by deplumbing that positive Hopf band.*
2. *If $F$ contains a negative Hopf band, then $(Y, F)$ supports an overtwisted contact structure.*
3. *(Yamamoto) If $(Y, F)$ admits a Stallings twist, then $(Y, F)$ supports an overtwisted contact structure.*

## Theorem (Baker-M.)

*A fibered Montesinos knot that supports the tight contact structure is isotopic to either*



$M(\frac{-d_1}{2d_1+1}, \frac{-d_2}{2d_2+1}, \ldots, \frac{-d_r}{2d_r+1} | 1)$



$M(\frac{-m_1}{m_1+1}, \frac{-m_2}{m_2+1}, \ldots, \frac{-m_r}{m_r+1} | 2)$

Figure: Left: odd type. Right: even type.

*and its fiber is obtained from the disk by a sequence of Hopf plumbings.*

# Odd case

$$K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_r}{\alpha_r} \mid e\right)$$

Each $\alpha_i/\beta_i$ has a strict continued fraction:

$$S_i = [2a_1^{(i)}, b_1^{(i)}, \ldots, 2a_{q_i}^{(i)}, b_{q_i}^{(i)}]$$

Hirasawa-Mursagi $\Rightarrow$ strong restrictions on $e$, $S_i$, for all $i$.

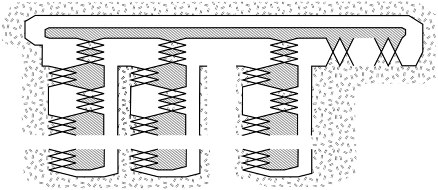


Figure: This image borrowed from Hirasawa-Murasugi.

# Odd case

- Repeatedly apply the Contact Structures Lemma, parts 2 & 3 to identify negative Hopf bands and/or Stallings twists.
- Cull these knots because they support an overtwisted contact structure.

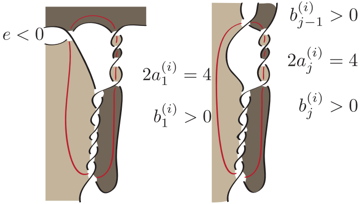
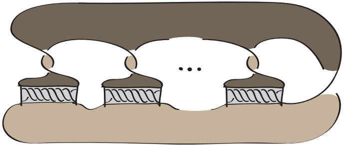



Figure: Finding negative Hopf bands in $F$.

# Odd case

- Odd fibered Montesinos knots without a $H^-$ remain.
- Successively deplumb $H^+$ until a single $H^+$ remains.
- These knots support the tight contact structure.




$M(\frac{-d_1}{2d_1+1}, \frac{-d_2}{2d_2+1}, \ldots, \frac{-d_r}{2d_r+1} \,|\, 1)$

# Determinant-genus inequality

**Lemma**

*Let $K$ be an odd fibered Montesinos knot supporting the tight contact structure. Then $\det(K) > 2g(K) + 1$ unless $K = M(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}|1)$.*

For any $K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2,} \ldots, \frac{\beta_r}{\alpha_r} \,|\, e\right)$,

$$\det(K) = |H_1(\Sigma_2(S^3, K); \mathbb{Z})| = \left|\prod_{i=1}^{r} \alpha_i \left(e + \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i}\right)\right|.$$

For odd, fibered Montesinos knots,

$$g(K) = \frac{1}{2}\left(\sum_{i=1}^{r} b^{(i)} + |e| - 1\right)$$

We verify $det(K) > 2g(K) + 1$ for such knots.

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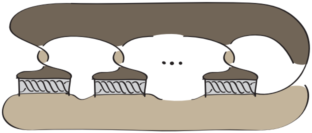
Finally, $K = M(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}|1)$ is the knot $10_{145}$. Since

$$\Delta_{10_{145}(t)} = t^2 + t - 3 + t^{-1} + t^{-2},$$

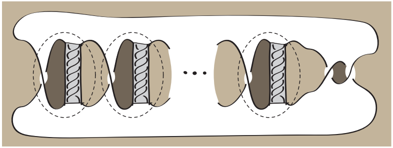
no odd fibered Montesinos knot admits an L-space surgery.

# Even case

Following similar techniques, pare down to the subfamily of fibered, even Montesinos knots which support the tight contact structure:



$M(\frac{-d_1}{2d_1+1}, \frac{-d_2}{2d_2+1}, \ldots, \frac{-d_r}{2d_r+1}|1)$



$M(\frac{-m_1}{m_1+1}, \frac{-m_2}{m_2+1}, \ldots, \frac{-m_r}{m_r+1}|2)$

## Lemma

$M(\frac{-m_1}{m_1+1}, \ldots, \frac{-m_r}{m_r+1}|2)$ *are isotopic to pretzel links.*

# Pretzel knots

**Theorem (Lidman-M.)**

*Let $K$ be a pretzel knot with any number of tangles. Then $K$ admits an L-space surgery if and only if*

1. $K \simeq T(2, 2n+1)$, $n \geq 0$, *or*
2. $K \simeq \pm(-2, 3, 2n+1)$, $n \geq 0$.

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### Remark

*These two families were previously known to admit L-space surgeries*

It remains to show no other pretzel knot qualifies.

# Pretzel knots

(Gabai 1980s): Classification of fibered pretzel links.

Two techniques to obstruct $K$ from admitting an L-space surgery.

- determinant-genus inequality
- $\Delta_K(t)$ obstructions using the Kauffman state sum:

$$\Delta_K(T) = \sum_{\mathbf{x} \in \mathcal{S}} (-1)^{M(\mathbf{x})} T^{A(\mathbf{x})}$$

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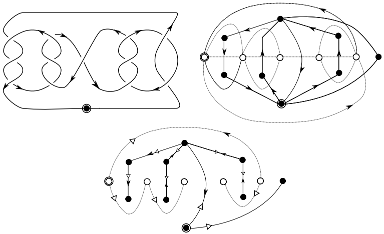
Kauffman state sum computations make use of the existence of essential Conway spheres in the complements of large pretzel knots.



# Essential $n$–string tangle decompositions

**Definition**

$K \subset S^3$ has an *essential $n$–string tangle decomposition* if $\exists$ embedded sphere $Q$ such that $Q \pitchfork K = \{2n \text{ pts}\}$ and where $Q - \partial\mathcal{N}(K)$ is essential in $S^3 - \mathcal{N}(K)$.

**Theorem (Krcatovich)**

*L-space knots are 1-string prime.*

**Conjecture (Lidman-M.)**

*L-space knots are 2-string prime.*

# More Questions

**Question**

*Can an lens space knot have an essential Conway sphere?*

Partial answer: Amongst arborescent knots, **no** (Wu).

**Question**

*Do L-space knots have any nontrivial mutants?*
*Are L-space knots n-string prime?*

Goals:

- Use contact geometry to study $Q \cap F_t$ and apply it to questions about mutation and knot Floer homology.
- Study the tunnel number of L-space knots.

Thank you!