Pretzel knots admitting L-space surgeries

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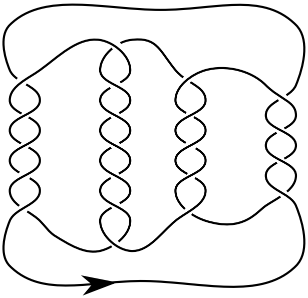
Pretzel knots

$K \subset S^3$.

Pretzel knot $(n_1, \ldots, n_k)$,
$n_i \in \mathbb{Z}$.



$K = (5, -7, 5, -4)$

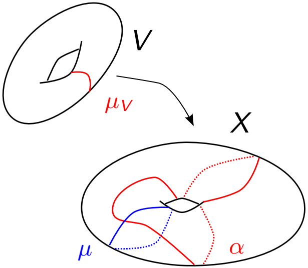
A familiar construction: Dehn surgery

$X = S^3 - \nu(K)$
$H_1(X) =< [\mu] >$
$H_1(\partial X) =< [\mu], [\lambda] >^{ab}$

$V \to X$

$\mu_V \mapsto \alpha$

$S^3_{p/q}(K)$ is called (p, q)-**Dehn surgery** along K.



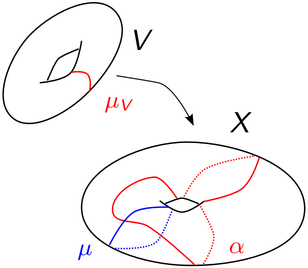
$\alpha = p\mu + q\lambda$

A familiar construction: Dehn surgery

Exercise:

$H_1(S^3_{p/q}(K)) = \mathbb{Z}/p\mathbb{Z}$.

Lens spaces arise from this construction when K is the unknot.



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A longstanding question

Question

Which knots admit lens space surgeries?

A bit of history

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1971 (Moser) some torus knots.

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1990 (Berge) many examples.

Cyclic Surgery Theorem (CGLS) $+$ Berge's construction $=$ 'The Berge Conjecture.'

Knot Floer homology

(Ozsváth-Szabó, Rasmussen):

$$\begin{aligned} K \subset M &\rightsquigarrow \cdots \subset \mathcal{F}_{i-1}C \subset \mathcal{F}_i C \subset \dots \\ &\rightsquigarrow H_*(\mathcal{F}_i C / \mathcal{F}_{i-1} C) \end{aligned}$$

For me:

- $M = S^3$.
- Coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.
- Simplest version.
- $(m, s) \in \mathbb{Z} \oplus \mathbb{Z}$.

$$\begin{aligned} \widehat{\mathrm{HFK}}(K) &= \bigoplus_{m,s} \widehat{\mathrm{HFK}}_m(K, s). \\ \Delta_K(t) &= \sum_s \chi(\widehat{\mathrm{HFK}}(K, s)) \cdot t^s \end{aligned}$$

L-spaces

Fact:

$$\text{rank}\,\widehat{HF}(L(p,q)) = p = |H_1(L(p,q))|.$$

A $\mathbb{Q}HS^3$ Y is an **L-space** if

$$|H_1(Y;\mathbb{Z})| = \text{rank}\,\widehat{HF}(Y).$$

Examples: S^3, all lens spaces, 3-manifolds with finite π_1.

Motivating question recast

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becomes

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L-space surgeries

Theorem (Ozsváth-Szabó)

If K admits an L-space surgery, then for all $s \in \mathbb{Z}$,

$$\widehat{HFK}(K, s) \cong \mathbb{F} \text{ or } 0.$$

(and some other conditions on Maslov grading).

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Corollary

If K admits an L-space surgery, $|a_i| \leq 1$ for all coefficients a_i of $\Delta_K(t)$.

Pretzel knots with L-space surgeries

Theorem (Lidman-M)

Let K be a pretzel knot with any number of tangles. Then K admits an L-space surgery if and only if

1. $K \simeq T(2, 2n+1)$, $n \in \mathbb{Z}$, *or*
2. $K \simeq \pm(-2, 3, q)$, $q \geq 1 \in \mathbb{Z}$ *odd.*

Remark

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5. $(-2, 3, 9)$*: has two finite non-cyclic surgeries (Bleiler-Hodgson).*

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6. $(-2, 3, q)$, $q > 11$ *odd:* $S^3_{2q+4}(K)$ *is a Seifert fibered L-space (Ozsváth-Szabó).*

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Theorem (Ni, Ghiggini)

K is fibered if and only if $\widehat{\mathrm{HFK}}(K, g(K)) \cong \mathbb{F}$.

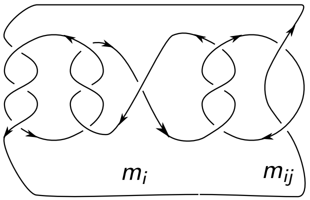
Theorem (Ozsváth-Szabó, Crowell-Murasugi)

An alternating knot admits an L-space surgery if and only if $K \simeq T(2, 2n+1)$, some $n \in \mathbb{Z}$.

Thus we only consider non-alternating, fibered pretzel knots.

Fibered pretzel links

Classified by Gabai in the 1980s.



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2. both m_i and m_{ij}.
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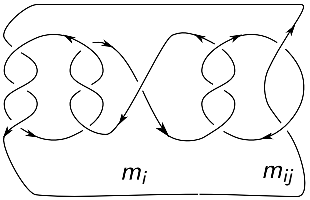

Three types:

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2. both m_i and m_{ij}.
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Within each type, we use two basic techniques to obstruct K from admitting an L-space surgery.

1. Show $det(K)$ is sufficiently large.
2. Compute coefficients of $\Delta_K(t)$ using the Kauffman state sum decomposition.

(1) Determinant condition

Lemma

If $\det(K) > 2g(K) + 1$, *then* K *is not an L-space knot.*

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If $\det(K) > 2g(K) + 1$, *then* K *is not an L-space knot.*

Proof.

If K is an L-space knot, then $|a_s| \leq 1 \; \forall s$. Then,

$$\det(K) = |\Delta_K(-1)| \leq \sum_s |a_s| \leq 2g(K) + 1.$$

□

(2) A state sum for $\Delta_K(t)$

Theorem (Kauffman)

The Alexander polynomial admits a state sum formula

$$\Delta_K(T) = \sum_{\mathbf{x} \in \mathcal{S}} (-1)^{M(\mathbf{x})} T^{A(\mathbf{x})},$$

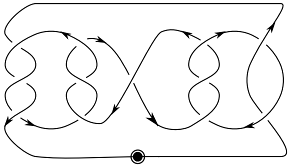
where $\mathcal{S}$ is the set of Kauffman states of a decorated knot diagram, and A and M are maps

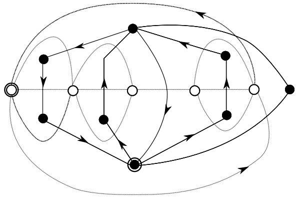
$$A : \mathcal{S} \to \mathbb{Z}, \quad M : \mathcal{S} \to \mathbb{Z}.$$

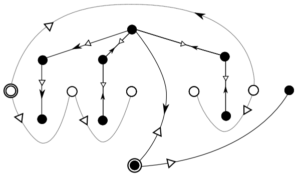
Kauffman states

Let G_B and G_W be the black and white graphs associated with a checkerboard coloring of a decorated knot projection.

$$\mathcal{S} \longleftrightarrow \{ \text{ spanning trees of } G_B \}$$







Basic idea

Apply mirroring, isotopy, and mutation to K until K^τ admits a diagram with a unique state $\mathbf{x}_0$ s.t.

$$A(\mathbf{x}_0) = -g(K^\tau).$$

Use $\mathbf{x}_0$ to count states $\mathbf{x}$ such that

$$A(\mathbf{x}) = -g(K^\tau) + 1,$$

and calculate $M(\mathbf{x})$ relative to $M(\mathbf{x}_0)$. Then show

$$|a_{-g(K^\tau+1)})| = \sum_{A(\mathbf{x})=-g(K^\tau)+1} (-1)^{M(\mathbf{x})} > 1.$$

Because the Alexander polynomial is preserved by mutation, obtain $|a_s| > 1$ for some coefficient a_s of $\Delta_K(t)$. □

Remarks

Corollary (Ichihara-Jong, Lidman-M.)

The only nontrivial pretzel knots admitting nontrivial finite surgeries are $\pm(-2, 3, q)$ *for* $q = 1, 3, 5, 7, 9$.

Observation

Up to mirroring, $\exists!$ *fibered pretzel knot with the Alexander polynomial of an L-space knot, but no L-space surgery. This is*

$$K = (3, -5, 3, -2).$$

Question

Observe that no pretzel knot admitting an L-space surgery contains an essential Conway sphere in its complement.

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Can an L-space knot have an essential Conway sphere?

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Do L-space knots have any nontrivial mutants?

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Answer: **No** (Wu).

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Question

Can the knot Floer complex "see" essential Conway spheres?

Thank you!

More info

Conjecturally, the total rank of knot Floer homology is invariant under (genus two) mutation. Evidence:

1. All knots through 12 crossings.
2. Many computed examples of 13 and 14 crossings.
3. Related conjecture of Baldwin-Levine about δ-graded $\widehat{\mathrm{HFK}}$.
4. And:

Theorem (M. and Starkston)

There exist infinitely many genus two mutant pairs K_n, K_n^τ with distinct knot Floer homology of the same total dimension.

Assuming total rank is preserved under mutation:

If true, then given an L-space knot K, and any mutant K^τ of K,

$$\widehat{\mathrm{HFK}}_m(K,s) \cong \widehat{\mathrm{HFK}}_m(K^\tau,s).$$

This suggests that $K \simeq K^\tau$, i.e., that L-space knots admit only trivial mutations.

The red-herring pretzel $(3, -5, 3, -2)$

	A	-3	-2	-1	0	1	2	3
M								
4								$\mathbb{F}$
3							$\mathbb{F}^3$	
2						$\mathbb{F}^4$	$\mathbb{F}^2$	
1					$\mathbb{F}^3$	$\mathbb{F}^4$		
0				$\mathbb{F}^4$	$\mathbb{F}^4$			
-1			$\mathbb{F}^3$	$\mathbb{F}^4$				
-2		$\mathbb{F}$	$\mathbb{F}^2$					
$\Delta_K(t)$		1	-1	0	1	0	-1	1