Montesinos knots, Hopf plumbings and L-space surgeries

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A longstanding question

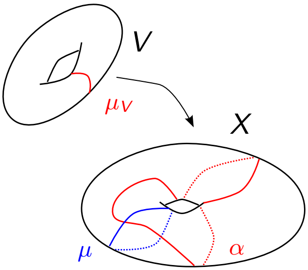
Which knots admit lens space surgeries?

1971 (Moser)

1977 (Bailey-Rolfsen)

1980 (Fintushel-Stern)

1990 (Berge)



$\alpha = p\mu + q\lambda$

Cyclic Surgery Theorem (CGLS) + Berge's construction
= "The Berge Conjecture."

L-spaces

(Ozsváth-Szabó, Rasmussen): Knot Floer homology.

$$K \subset Y \longrightarrow \cdots \subset \mathcal{F}_{i-1}C \subset \mathcal{F}_i C \subset \dots$$
$$\downarrow$$
$$H_*(\mathcal{F}_i C / \mathcal{F}_{i-1} C)$$
$$||$$
$$\widehat{\mathrm{HFK}}(K) = \bigoplus_{m,s} \widehat{\mathrm{HFK}}_m(S^3, K, s).$$

- $\Delta_K(t) = \sum_s \chi(\widehat{\mathrm{HFK}}(K,s)) \cdot t^s$
- A $\mathbb{Q}HS^3$ Y is an **L-space** if $|H_1(Y;\mathbb{Z})| = \mathrm{rank}\, \widehat{HF}(Y)$.
  Ex: S^3, all lens spaces, 3-manifolds with finite π_1.

Motivating question revisited

Question

Which knots admit lens space surgeries?

becomes

Question

Which knots admit L-space surgeries?

L-space surgery obstructions

Theorem (Ozsváth-Szabó)

If K admits an L-space surgery, then for all $s \in \mathbb{Z}$, $\widehat{HFK}(K, s) \cong \mathbb{F}$ or 0 (and some other conditions on Maslov grading).

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If $det(K) > 2g(K) + 1$, then K is not an L-space knot.

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Proof.

If K is an L-space knot, then $|a_s| \leq 1$ $\forall$ coefficients a_s of $\Delta_K(t)$. Then,

$$\det(K) = |\Delta_K(-1)| \leq \sum_s |a_s| \leq 2g(K)+1.$$

□

More geometric obstructions

Theorem (Ni, Ghiggini)

K is fibered if and only if $\widehat{\mathrm{HFK}}(K, g(K)) \cong \mathbb{F}$.

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Theorem (Hedden)

An L-space knot K supports the tight contact structure; equivalently, an L-space knot is strongly quasipositive.

Classification theorem

Theorem (Baker-M.)

Among the Montesinos knots, the only L-space knots are

- *the pretzel knots* $P(-2,3,2n+1)$ *for* $n \geq 0$,
- *and the torus knots* $T(2,2n+1)$ *for* $n \geq 0$.

Montesinos knots

$$K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \dots, \frac{\beta_r}{\alpha_r} \,\middle|\, e\right)$$

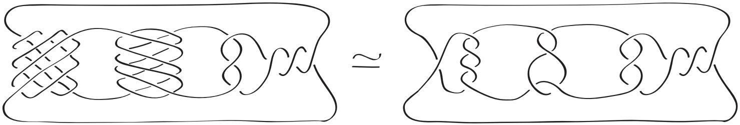

Figure: $M(\frac{3}{4}, -\frac{2}{5}, \frac{1}{3}|3)$.

Where $\alpha_i, \beta_i, e \in \mathbb{Z}$ and $\alpha_i > 1$, $|\beta_i| < \alpha_i$, and $\gcd(\alpha_i, \beta_i) = 1$.

Ingredients for proof

We need only consider fibered, non-alternating Montesinos knots,

$$K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2,} \dots, \frac{\beta_r}{\alpha_r} \mid e\right)$$

and we assume $r \geq 3$, because $r \leq 2$ implies K is a two-bridge link.

Theorem (Ozsváth-Szabó)

An alternating knot admits an L-space surgery if and only if $K \simeq T(2, 2n+1)$, some $n \in \mathbb{Z}$.

Fibered Montesinos knots

(Hirasawa-Murasugi): Classified fibered Montesinos knots with their fibers. For $K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_{2,}} \ldots, \frac{\beta_r}{\alpha_r} \mid e\right)$,

$$\frac{\beta_i}{\alpha_i} = \cfrac{1}{x_1 - \cfrac{1}{x_2 - \cfrac{1}{\ddots - \cfrac{1}{x_m}}}}$$

$$S_i := [x_1, \ldots, x_m]$$

have two cases of S_i:

1. α_i are all odd $\rightsquigarrow$ strict continued fractions.
2. α_1 is even, α_i is odd for $i > 1$ $\rightsquigarrow$ even continued fractions.

Example: odd case

Each β_i/α_i has a strict continued fraction:

$$S_i = [2a_1^{(i)}, b_1^{(i)}, \ldots, 2a_{q_i}^{(i)}, b_{q_i}^{(i)}]$$

Hirasawa-Mursagi give strong restrictions on e, $S_1, \ldots, S_m$ when M is fibered.

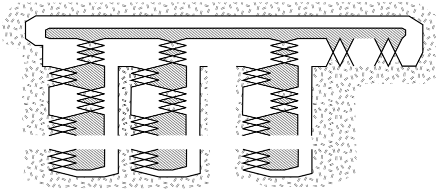

Figure: Image of odd-type Seifert surface borrowed from Hirasawa-Murasugi.

Open books for three-manifolds

(F, ϕ) —an open book for closed 3-manifold Y.

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ξ —a contact structure on Y.

- Locally, $\ker \alpha$, $\alpha \wedge d\alpha \neq 0$
- (Thurston-Winkelnkemper - 1975)
  Every (F, ϕ) induces a contact structure.
- (Giroux - 2000)
  $\{$or. ξ on $Y\}/$ isotopy $\longleftrightarrow \{(F, \phi)$ for $Y\}$ / positive stabilization

Plumbings of Hopf bands

Hopf links:

- $L^+ = \{(z_1, z_2) \in S^3 \subset \mathbb{C}^2 \mid z_1 z_2 = 0\}$.
- $L^- = \{(z_1, z_2) \in S^3 \subset \mathbb{C}^2 \mid z_1 \overline{z_2} = 0\}$.

Pos/neg (de)stabilization $\leftrightarrow$ (de)plumbing of pos/neg Hopf bands.

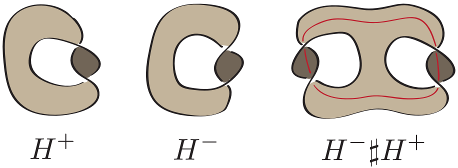

Figure: The connected sum of a positive and negative Hopf band.

Lemma (Contact Structures Lemma)

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2. *(Yamamoto):*
   If F contains a twisting loop, then $\xi_{(F,\phi)}$ is overtwisted.
3. *(Giroux):*
   If $F \supset H_+$ and

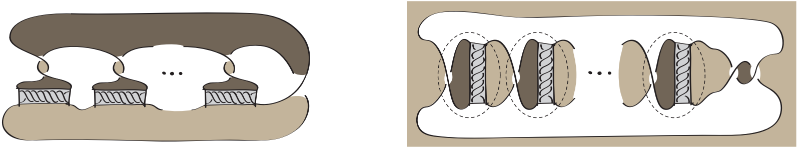
   $$(F, \phi) = (F', \phi') * (H_+, \pi^+)$$

   then

   $$\xi_{(F,\phi)} \simeq \xi_{(F',\phi')}.$$

Theorem (Baker-M.)

A fibered Montesinos knot that supports the tight contact structure is isotopic to either



$M(\frac{-d_1}{2d_1+1}, \frac{-d_2}{2d_2+1}, \ldots, \frac{-d_r}{2d_r+1} | 1)$ $M(\frac{-m_1}{m_1+1}, \frac{-m_2}{m_2+1}, \ldots, \frac{-m_r}{m_r+1} | 2)$

Figure: Left: odd type. Right: even type.

and its fiber is obtained from the disk by a sequence of Hopf plumbings.

Odd case

- Repeatedly apply the Contact Structures Lemma, parts 1 & 2 to identify negative Hopf bands and/or twisting loops.
- Cull these knots because they support an overtwisted contact structure.

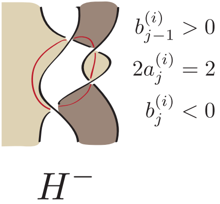
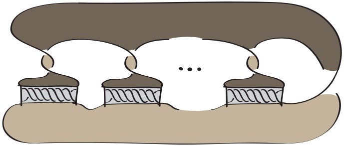


Figure: Finding negative Hopf bands in F.

Odd case

- Odd fibered Montesinos knots without a H^- remain.
- Successively deplumb H^+ until a single H^+ remains.
- These knots support the tight contact structure.



$M(\frac{-d_1}{2d_1+1}, \frac{-d_2}{2d_2+1}, \ldots, \frac{-d_r}{2d_r+1} \big| 1)$

Determinant-genus inequality

Lemma

Let K be an odd fibered Montesinos knot supporting the tight contact structure. Then $\det(K) > 2g(K) + 1$ unless $K = M(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}|1)$.

For any $K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_r}{\alpha_r} \,|\, e\right)$,

$$\det(K) = |H_1(\Sigma_2(S^3, K); \mathbb{Z})| = \left| \prod_{i=1}^{r} \alpha_i \left(e + \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i} \right) \right|.$$

For odd, fibered Montesinos knots,

$$g(K) = \frac{1}{2}\left(\sum_{i=1}^{r} b^{(i)} + |e| - 1\right)$$

We verify $det(K) > 2g(K) + 1$ for such knots.

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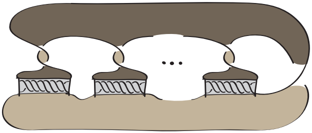
Finally, $K = M(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}|1)$ is the knot 10_{145}. Since

$$\Delta_{10_{145}(t)} = t^2 + t - 3 + t^{-1} + t^{-2},$$

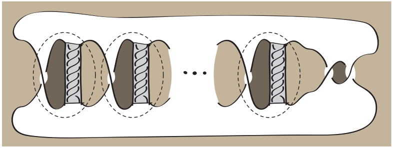
no odd fibered Montesinos knot admits an L-space surgery.

Even case

Similarly, pare down to the subfamily of fibered, even Montesinos knots which support the tight contact structure:



$M(\frac{-d_1}{2d_1+1}, \frac{-d_2}{2d_2+1}, \ldots, \frac{-d_r}{2d_r+1}|1)$



$M(\frac{-m_1}{m_1+1}, \frac{-m_2}{m_2+1}, \ldots, \frac{-m_r}{m_r+1}|2)$

Lemma

$M(\frac{-m_1}{m_1+1}, \ldots, \frac{-m_r}{m_r+1}|2)$ *are isotopic to pretzel links.*

Pretzel knots

Theorem (Lidman-M.)

A pretzel knot admits an L-space surgery if and only if $K \simeq T(2, 2n+1)$, $n \geq 0$, *or* $K \simeq \pm(-2, 3, 2n+1)$, $n \geq 0$.

- Gabai's classification of fibered pretzel links.
- determinant-genus inequality
- $\Delta_K(t)$ obstructions using the Kauffman state sum:

$$\Delta_K(T) = \sum_{\mathbf{x} \in \mathcal{S}} (-1)^{M(\mathbf{x})} T^{A(\mathbf{x})}$$

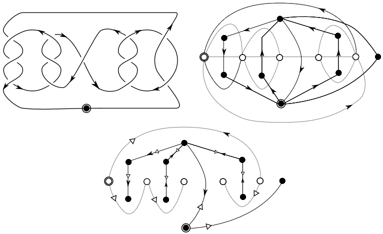


Figure: Computations use existence of essential Conway spheres.

Essential n–string tangle decompositions

Definition

$K \subset S^3$ has an *essential n–string tangle decomposition* if $\exists$ embedded sphere Q such that $Q \pitchfork K = \{2n \text{ pts}\}$ and where $Q - \partial\mathcal{N}(K)$ is essential in $S^3 - \mathcal{N}(K)$.

Theorem (Krcatovich)

L-space knots are 1-string prime.

Conjecture (Lidman-M.)

L-space knots are 2-string prime.

Remark: (Wu) $\Rightarrow$ Amongst arborescent knots, a lens space knot cannot have an essential Conway sphere.

Braided satellites

- (Hayahsi-Matsuda-Ozawa): If a braided satellite knot has an essential tangle decomposition, then its companion has an essential tangle decomposition, too.
- (Hom-Lidman-Vafaee): An L-space knot that is a Berge-Gabai satellite knot must have an L-space knot as its companion.

If there exists a Berge-Gabai L-space knot with an essential tangle decomposition, its companion will also be an L-space knot with an essential tangle decomposition.

Tunnel number

What can we say about tunnel number?

- Many L-space knots have tunnel number one.
- Tunnel number one knots are n–string prime. (Gordon-Reid)
- For all N, there exists an L-space knot with tunnel number N. (Baker-M.)
- There exists a hyperbolic L-space knot with tunnel number two. (Motegi).

Thank you!