ADJACENCY OF THREE-MANIFOLDS AND BRUNNIAN LINKS

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Abstract. We introduce the notion of adjacency in three-manifolds. A three-manifold Y is n-adjacent to another three-manifold Z if there exists an n-component link in Y and surgery slopes for that link such that performing Dehn surgery along any nonempty sublink yields Z. We characterize adjacencies from three-manifolds to the three-sphere, providing an analogy to Askitas and Kalfagianni's results on n-adjacency in knots.

1. Introduction

A knot K is said to be n-adjacent to a knot K' if there exists a diagram of K containing a set of n crossings such that changing any nonempty subset of them yields a diagram of K'. Any knot that is adjacent to the unknot is, of course, unknotting number one, but the condition is much more restrictive. For example, nontrivial knots which are n-adjacent to the unknot for $n \geq 3$ have trivial Alexander polynomial, are non-fibered, non-alternating, and have vanishing Vassiliev invariants of degree less than $2n-1$. These restrictions on their invariants are shown by Askitas-Kalfagianni [AK02] to result from a diagrammatic characterization of n-adjacent knots, $n \geq 3$, as those constructed from certain spatial chord diagrams called Brunnian-Suzuki graphs [AK02, Theorem 4.4]. A Suzuki graph is given by a collection of weighted, embedded arcs along an unknotted circle in S^3, where the arcs describe a pattern along which a sequence of bandings and clasps converts the graph into a knot.

In this article, we generalize the concept of n-adjacency from knots to three-manifolds. We say that a closed, oriented three-manifold Y is *integrally n-adjacent* to another three-manifold Z if there exists an n-component link L and integral multi-slope p of the link such that performing Dehn surgery along any nonempty subset of L yields Z. The triple realizing the adjacency will be denoted (Y, L, p). *Rational adjacency*, denoted (Y, L, α), is defined similarly where rational surgeries are permitted. In an analogy to Askitas-Kalfagianni, we characterize all n-adjacencies to the three-sphere as those arising from particular Dehn surgeries along Brunnian-like links. This recovers Askitas-Kalfagianni's diagrammatic characterization of knots adjacent to the unknot, and we similarly obtain a statement on the finite type invariants of three-manifolds.

Given a link in a three-manifold, and a choice of surgery slopes, the core curves of the surgery solid tori produce a new link in the surgered manifold. We call this the *core* or *dual* of the surgery. Using dual links, it is in fact quite easy to construct examples of adjacent manifolds for all n: Let J be a Brunnian link in S^3, and perform $(\pm 1, \dots, \pm 1)$-surgery on J. This yields a homology sphere Y that is n-adjacent to the three-sphere via

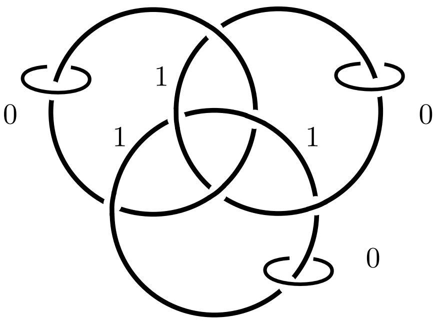

FIGURE 1. The Poincaré homology sphere results from $(1,1,1)$-surgery on the Borromean rings. The meridians with surgery slopes $(0,0,0)$ realize a 3-adjacency of the Poincaré homology sphere to S^3.

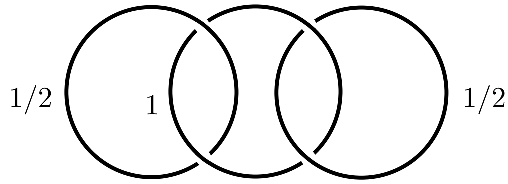

FIGURE 2. This three-component Hopf-Brunnian link J admits a $(1/2,1,1/2)$-rational surgery to $-L(3,1)$. The core of J after surgery is a link L realizing the 3-adjacency of $-L(3,1)$ to S^3.

the dual link. The homology sphere will be distinct from S^3 provided that J is a nontrivial link. For a concrete example, $(1,1,1)$-surgery on the Borromean rings produces the Poincaré homology sphere, which is 3-adjacent to S^3, as shown in Figure 1. In fact, we will see shortly that all integral n-adjacencies for $n \geq 3$ arise from surgery on Brunnian links.

Next, consider the example in Figure 2. An exercise in Kirby calculus shows that $(1/2,1,1/2)$-surgery yields $-L(3,1)$ and induced surgery on any proper sublink yields S^3. This surgery description will demonstrate that $-L(3,1)$ is 3-adjacent to S^3. Thus, Hopf links can and do appear in the characterization of rational adjacencies. We define the family of *Hopf-Brunnian* links as follows. An n-component link is Hopf-Brunnian if all $n-1$ component sublinks are split unions of Hopf links and unknots. We prove:

Theorem 1.1. *The triple (Y, L, α) realizes an n-adjacency to S^3 with surgery core J if and only if J is Hopf-Brunnian, the dual surgery slopes of J are of the form $1/k_i$, $k_i \in \mathbb{Z}^*$, and any proper Hopf sublink of J must have surgery slopes $\pm(1,1/2)$ or $\pm(1/2,1)$.*

Corollary 1.2. *The triple (Y, L, p) realizes an integral n-adjacency to S^3 with surgery core J if and only if J is Brunnian and the dual surgery slopes of J are ± 1 (signs need not be consistent).*

Corollary 1.3. *If Y is integrally n-adjacent to S^3 for $n \geq 3$, or rationally n-adjacent to S^3 for $n \geq 4$, then Y is an integer homology sphere.*

Note that in the case $n = 2$, any $(1/k_1, 1/k_2), k_i \in \mathbb{Z}^*$, surgery along any link $J_1 \cup J_2$ of unknotted components in S^3 will yield a manifold adjacent to the three-sphere. In order to prove Theorem 1.1, we give a stronger characterization for self-adjacencies from the three-sphere to itself.

Proposition 3.2. *The triple (S^3, J, α) realizes an n-adjacency to S^3 if and only if J itself is a split union of Hopf links and unknots, all slopes $\alpha_i = 1/k_i$, where $k_i \in \mathbb{Z}^*$, and the surgery slopes of Hopf components are either $\pm(1, 1/2)$ or $\pm(1/2, 1)$.*

Note that the requirement that J itself is a split union of Hopf links and unknots is stronger than requiring J be Hopf-Brunnian.

Using Proposition 3.2, we may now prove Theorem 1.1.

Proof of Theorem 1.1. In Proposition 2.2 below, we show that if (Y, L, α) realizes a rational n-adjacency to S^3, then the core of the surgery is an n-component link J in S^3 and performing α-framed surgery on $\cup_{i \in I} L_i$, for any $I \subset \{1, \ldots, n\}$, yields the same result as performing surgery on $\cup_{i \in [n]-I} J_i$ with the corresponding dual slopes (see Section 2.1 for more details). In particular, surgery on every proper sublink of J in S^3 gives back S^3. By Proposition 3.2, every proper sublink of J is a split union of Hopf links and an unlink. The Hopf pairs have surgery coefficients $\pm(1, 1/2)$ and split unknotted components have surgery coefficients $1/k$, $k \neq 0$. ☐

Let us return to n-adjacency in knots. Using Theorem 1.1, we are now able to recover Askitas-Kalfagianni's characterization of knots which are n-adjacent to the unknot [AK02, Theorem 4.4].

Theorem 1.4 (Askitas-Kalfagianni). *Let K be n-adjacent to the unknot for $n \geq 3$. Then K is the realization of a Brunnian-Suzuki n-graph.*

Proof. Let $n \geq 3$, and let K be n-adjacent to the unknot U. The collection of unknotting arcs lifts to a strongly invertible link $L = L_1 \cup \ldots \cup L_n$ in the double cover of S^3 branched over K, which we will call Y. Likewise, there is a corresponding collection of 'knotting' arcs $\gamma_1, \ldots, \gamma_n$ from U to K, which lifts to a strongly invertible link J in S^3. The Montesinos trick [Mon75] provides a half-integral multi-slope β on J such that β_i-surgery on J_i corresponds to the associated crossing change downstairs on γ_i.

Applying crossing changes associated to any subset of crossing arcs $\cup_{i \in I} \gamma_i$ along U yields the same result as applying the complementary $[n] - I$ crossing changes in K. By the n-adjacency of K, this produces the unknot for any proper subset $I \subset \{1, \cdots, n\}$. At the level of the branched double cover, we see that surgery on every proper sublink of J produces S^3. (See Proposition 2.2 below.) We now apply Proposition 3.2 to the $(n-1)$-component sublinks of J. Since $n \geq 3$ and none of the surgery slopes are ± 1 (because they are all half-integral), all of the pairwise linking numbers of J must be zero and the proper sublinks are unlinks, *i.e.* J is Brunnian, and each β_i is $\pm 1/2$.

Consider now the union of the arcs $\gamma_1, \ldots, \gamma_n$ together with the unknot U. Again by the Montesinos trick [Mon75], an arc with weight (w, z) in the terminology of [AK02, Section 3] realizes a surgery in the branched double cover with slope $w + \frac{z}{2}$. The theorem will follow from the following claim. ☐

Claim 1. *The graph* $G = U \cup \bigcup_{i=1}^{n} \gamma_i$ *is a Brunnian Suzuki* n*-graph.*

Proof of claim. Notice that no pair of arcs γ_i and γ_j have endpoints interleaved along the unknot because they would lift to a link $J_i \cup J_j$ of nonzero linking number, a contradiction. This means that G is *admissible*, in the terminology of [AK02, Definition 3.2]. For any proper subset $I \subset \{1, \ldots, n\}$ of components of $J_1 \cup \ldots \cup J_n$, the quotient under τ is a proper subset of the γ arcs. Because each subset of J is an unlink, and there is a unique strong inversion τ on the unlink [KT80], these descend to arcs embedded disjointly in the spanning disk for the unknot. Morever, because the surgery slopes on each J_i are $\pm 1/2$, these descend to weighted arcs of the form $(0, \pm 1)$. Thus, subgraphs of G with $n-1$ arcs are standard. The graph G is therefore a Brunnian-Suzuki n-graph, as in [AK02, Definition 3.4]. ☐

Note that Theorem 1.4 is the key ingredient in the claimed vanishing of the Vassiliev invariants of a knot n-adjacent to the unknot, as mentioned above.

In analogy with the work of Askitas-Kalfagianni, we are also able to apply Theorem 1.1 more generally to prove a vanishing result for finite-type invariants of homology spheres. (For a quick survey on finite-type invariants, see [Lin98].)

Corollary 1.5. *Let* Y *be a homology sphere which is integrally* n*-adjacent to* S^3*. Then all finite-type invariants of order less than* $2n-4$ *vanish.*

Proof. By Corollary 1.2, Y is obtained by surgery on an n-component Brunnian link. The required vanishing result is given by [Mei06, Theorem 1.1]. ☐

2. Dehn surgery

2.1. **The dual perspective.** Let Y be a closed, oriented three-manifold, and let $L = L_1 \cup ... \cup L_n$ denote a link in Y. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote a multi-slope on L. The notation $Y_\alpha(L)$ denotes the three-manifold obtained by performing α_i Dehn surgery along L_i for all $i = 1, \ldots, n$.

Definition 2.1. Consider the triple (Y, L, α), where Y is a closed, oriented 3-manifold, $L = L_1 \cup \ldots \cup L_n$ is a link in Y, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-slope on L. Let Z be a closed, oriented three-manifold. If $Y_{\alpha_I}(L_I) = Z$ for *any* nonempty subset I of $\{1, ..., n\}$, then (Y, L, α) *realizes* an n-adjacency to Z. We say that Y is *integrally* n-adjacent to Z if the multi-slopes are integral and *rationally* adjacent to Z otherwise.

Notice that n-adjacency is not a symmetric relation.

In order to circumvent the difficulty in describing surgeries in arbitrary three-manifolds, we take the following perspective.

Suppose that L is an n-component link in a three-manifold Y with a surgery to S^3 along the multi-slope α. Let J be the core of the surgery in S^3, and write $J = J_1 \cup \ldots \cup J_n$. Then, we will associate to J rational numbers $r = (r_1, \ldots, r_n)$ which describe how to "undo" the surgery performed on each J_i. To be more precise, in the exterior of L, we have two slopes η_i and α_i on the boundary torus coming from L_i: η_i is the meridian of L_i and Dehn filling along α_i corresponds to the non-trivial surgery we are going to do to get to S^3. Viewing J as a link in S^3, we still can view these slopes in the boundary of a neighborhood of J_i. Express $\eta_i = p_i\mu_i + q_i\lambda_i$, where μ_i, λ_i are the meridian and longitudes for J_i as a knot in S^3. (Note that $\mu_i = \alpha_i$.) We say that the slopes η_i and α_i are dual to each other. We calculate these in the following order:

(1) First, perform surgery on *all* components of L to get J in S^3, not just some of the components (which still produces S^3 if L is realizing an n-adjacency).

(2) Then, identify the slopes η_i with $r_i = p_i/q_i$.

Note that the adjacency is integral if and only if all p_i/q_i are integral. This is because $\Delta(\eta_i, \alpha_i) = \Delta(\mu_i, p_i\mu_i + q_i\lambda_i) = |q_i|$. In general, when discussing the dual link of a surgery to S^3, we will assume that it naturally inherits these rational surgery slopes in S^3 as above.

Now, the data (S^3, J, r) in S^3 actually recovers (Y, L, α). Performing surgery on all components of J gives Y and by construction L is the core of the surgery on J while the meridian of each J_i becomes the slope α_i. But, we can also recover sublinks in the following way. If we look at a sublink J' of J, without loss of generality, $J_1 \cup \ldots \cup J_k$, then surgery on J' produces the same manifold as surgery in Y on $L_{k+1} \cup \ldots \cup L_n$. And further, the core of surgery on J', a k-component link, is exactly the *image* of $L_1 \cup \ldots \cup L_k$ in the surgery on $L_{k+1} \cup \ldots \cup L_n$.

Now we return to the case that a rational surgery on L is realizing an n-adjacency from Y to S^3. Build (S^3, J, r) as discussed. The above paragraph can be reinterpreted as saying that doing the corresponding surgery on every proper sublink of J gives S^3. For the benefit of the reader, we summarize this discussion with the following proposition.

Proposition 2.2. *Let α be a multi-slope on an n-component link L in Y. Then (Y, L, α) realizes an n-adjacency to S^3 if and only if there exists a multi-slope β on a link J in S^3 such that:*

(1) *$S^3_\beta(J) = Y$;*

(2) *L is the core of the surgery on J;*

(3) *each α_i is the dual slope to β_i;*

(4) *surgery on every proper sublink of J yields S^3.*

Furthermore, the adjacency is integral if and only if β is integral.

2.2. **The linking of the dual curves.** In light of the dual perspective from Proposition 2.2, we want to understand the effects of surgery on links in S^3 whose sublinks also surger to S^3. The next lemma allows us to constrain the linking numbers and surgery coefficients for the dual link in S^3 arising from an n-adjacency.

Lemma 2.3. *Suppose that (S^3, J, α) realizes a 2-adjacency to S^3. Then either the linking number of J is zero, or the linking number is ± 1 and the surgery coefficients α_i are $\pm(1, 1/2)$ or $\pm(1/2, 1)$.*

Proof. Since surgery on each individual component of $J = J_1 \cup J_2$ produces S^3, an integer homology sphere, the surgery coefficient for J_i is of the form $1/q_i$. The linking matrix for the surgery presentation on J then gives

$$1 = \left| det \begin{pmatrix} 1 & q_2\ell \\ q_1\ell & 1 \end{pmatrix} \right| = |q_1 q_2 \ell^2 - 1|,$$

where ℓ is the linking number of J_1 and J_2 (say after choosing orientations of each component). Since $q_1, q_2 \neq 0$, we see that $|\ell| = 0$ or 1. If $|\ell| = 1$, then we must have that $(q_1, q_2) = \pm(1, 2)$ or $\pm(2, 1)$, as desired. $\square$

Proposition 2.4. *Suppose that J is an n-component link in S^3, with $n \geq 3$, and α is a multi-slope on J. If surgery on every proper sublink of J produces S^3, then all pairwise linking numbers are 0 or ± 1. If J_1 and J_2 are a pair of components with $|\ell k(J_1, J_2)| = 1$, then the slopes are $\pm(1, 1/2)$ or $\pm(1/2, 1)$. If $n = 3$ and $S^3_\alpha(J)$ is an integer homology sphere or $n \geq 4$, then each of J_1 and J_2 has linking number zero with all other components.*

Proof. Since $n \geq 3$, every two-component sublink of J with induced multi-slope from α provides a 2-adjacency from S^3 to itself. Therefore, by Lemma 2.3, the pairwise linking numbers are 0 or ± 1 and for the 2-component sublinks with linking number having absolute value 1, the surgery coefficients are $\pm(1, 1/2)$ or $\pm(1/2, 1)$.

Now suppose that $S^3_\alpha(J)$ is an integer homology sphere. It remains to consider the pairwise linking of J_1, J_2 with the other components. Let J_3 be another component. Then, we know that the associated surgery on $J_1 \cup J_2 \cup J_3$ produces $S^3_\alpha(J)$ if $n = 3$ and S^3 if $n > 3$. Either way, the result is an integer homology sphere. Orient J_1, J_2 such that the pairwise linking is 1, fix an orientation on J_3 and let ℓ_1, ℓ_2 be the linking numbers of J_3 with J_1, J_2 respectively. Without loss of generality, the surgery coefficients on J_1 and J_2 are 1 and 1/2 respectively. (Otherwise, rearrange the order of the components and/or mirror J and reverse the signs of α.) Suppose for contradiction that ℓ_1, ℓ_2 are not both zero.

The first case is that $\ell_1 \neq 0$. In this case, by applying the first part of the proposition to the pair (J_1, J_3), we see that $\ell_1 = 1$ and the surgery coefficient for J_3 must be 1/2. Applying the first part of the proposition to the pair (J_2, J_3), we see that J_2 and J_3 have linking number zero, since the pair of surgery coefficients is not $\pm(1/2, 1)$ or $\pm(1, 1/2)$.

In this case, the linking matrix for the 3-component surgery description computes the order of H_1 of the surgery on $J_1 \cup J_2 \cup J_3$ to be:

$$1 = \left| det \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right| = 3,$$

a contradiction.

The other case is that $\ell_1 = 0$, and so $\ell_2 = 1$. Now we see the surgery coefficients are 1 for J_1 and J_3, and $1/2$ for J_2. Then, we compute again

$$1 = \left| det \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \right| = 3,$$

another contradiction. This completes the proof. □

3. Self-adjacencies from S^3

In this section, we constrain the self-adjacencies from S^3 to itself. As a warm-up, we begin with a special case.

Proposition 3.1. *Suppose (S^3, J, α) realizes an n-adjacency to S^3 and the pairwise linking numbers of J vanish. Then J is the unlink and $\alpha_i = 1/k_i$ for $k_i \in \mathbb{Z}^*$ for all i.*

Proof. First, it is clear that $\alpha_i = 1/k_i$ for each i by a homological computation. We proceed by induction to show that J is the unlink. The case of $n = 1$ is handled by the knot complement theorem [GL89]. Next we do the case of a 2-component link, $J_1 \cup J_2$. From the $n = 1$ case, each component is unknotted. Since $1/k_2$-surgery on J_2 is S^3, the image of J_1 in $1/k_2$-surgery on J_2 has a surgery to S^3. Hence, the image of J_1 is also unknotted. In particular, the component J_2 in the complement of J_1 in S^3 is a knot in a solid torus which has a non-trivial solid torus surgery. By [Gab89], J_1 is contained in a ball or is a braid in the solid torus, so has non-zero winding number. However, because $\ell k(J_1, J_2) = 0$, it must be the case that J_1 is contained in a ball in the complement of J_2 in S^3, meaning J_1 and J_2 are unlinked. This completes the proof for $n = 2$ components.

For the inductive step, our hypothesis is that J is a Brunnian link and then we will deduce that J is in fact an unlink. This can be found in [GLLM22, Proposition 4.1], but for self-containedness, we present an elementary proof that does not rely on Heegaard Floer homology. Suppose the result is true for $(n-1)$-component links and that (S^3, J, α) realizes an n-adjacency to S^3. Then J is Brunnian, $J_1 \cup \ldots \cup J_{n-1}$ is an unlink, and so $(1/k_1, \ldots, 1/k_{n-1})$-surgery on $J_1 \cup \ldots \cup J_{n-1}$ gives S^3. Thus the image of J_n must be unknotted after this surgery. Hence, we see that $(1/k_1, \ldots, 1/k_{n-1}, 1/m)$-surgery on J gives S^3 for arbitrary m. For the sake of concreteness, fix $m = 5$.

We claim that the image of $J_1 \cup \ldots \cup J_{n-1}$ after performing $1/5$-surgery on J_n, denoted $K_1 \cup \ldots \cup K_{n-1}$, yields an $(n-1)$-component Brunnian link. To see that the $(n-1)$-component image link is Brunnian, note that all $(n-1)$-component sublinks of J are unlinks by our inductive hypothesis. In particular, $J_1 \cup \ldots \cup J_{n-2} \cup J_n$ is an unlink, hence $1/5$-surgery along J_n shows that $K_1 \cup \ldots \cup K_{n-2}$ is an unlink. A similar argument applies to the other $n-2$-component sublinks of $K_1 \cup \cdots \cup K_{n-1}$.

Thus, the link $K_1 \cup \cdots \cup K_{n-1}$ is a Brunnian link with a surgery to S^3, and so is an unlink by induction. In other words, $1/5$-surgery on J_n in the exterior of $J_1 \cup \ldots \cup J_{n-1}$ produces a reducible 3-manifold (the exterior of an $(n-1)$-component unlink). Yet, the distance between the trivial slope ∞ and $1/5$ is 5. A theorem of Gordon and Litherland [GL84, Theorem 1.1] implies that for any pair of reducible Dehn fillings on an irreducible manifold, the slopes have distance at most four. This is a contradiction. This implies that the exterior of J must be reducible. Hence, J is split. However, a split Brunnian link is an unlink. □

We now build on the previous proposition to complete our characterization of the self-adjacencies of S^3 promised in the introduction.

Proposition 3.2. *The triple (S^3, J, α) realizes an n-adjacency to S^3 if and only if J itself is a split union of Hopf links and unknots, all slopes $\alpha_i = 1/k_i$, where $k_i \in \mathbb{Z}^*$, and the surgery slopes of Hopf components are either $\pm(1, 1/2)$ or $\pm(1/2, 1)$.*

Proof. As in the proof of Proposition 3.1, the knot complement theorem implies that the components are unknotted and of course the surgery coefficients are of the form $1/k_i$.

We begin with the case of $n = 2$. Consider $J_1 \cup J_2$. The case that $\ell k(J_1, J_2) = 0$ follows from Proposition 3.1. By Lemma 2.3, we assume $\ell k(J_1, J_2) = 1$ and the surgery coefficients are $\pm(1, 1/2)$. Note that J_1 is unknotted in $1/2$-surgery on J_2, so J_2 can again be viewed as a knot in the solid torus with a solid torus surgery. Therefore, by [Gab89], J_2 is a braid in the complement of J_1 and the winding number is the linking number of J_1 and J_2. The only winding number 1 braid in the solid torus is the core. We see that $J_1 \cup J_2$ is a Hopf link.

Next, we handle the case of $n = 3$. If the pairwise linking numbers are zero, we appeal to Proposition 3.1. So, assume some pair of components J_1, J_2 have nonzero linking number. By Proposition 2.4, J_1, J_2 have surgery coefficients $\pm(1, 1/2)$ and linking number 1. Further, by Proposition 2.4, we have that J_3 is algebraically split from J_1 and J_2. By the $n = 2$ case of the proof, $J_1 \cup J_2$ must form a Hopf link. We also have that J_3 is an unknot that is geometrically split from J_1 and J_2 individually, but possibly not split from the link $J_1 \cup J_2$.

It remains to show that J_3 is in fact split from $J_1 \cup J_2$. Since $J_1 \cup J_3$ is an unlink, trivial surgery on J_2 produces the 2-component unlink $J_1 \cup J_3$. Also, the image of $J_1 \cup J_3$ under $1/2$-surgery on J_2 is a 2-component unlink because this image is a 2-component link, all of whose induced surgeries give S^3. We now appeal to [CGLS87, Corollary 2.4.7]. This states that if an irreducible three-manifold admits two reducible Dehn fillings along slopes of distance at least two on a torus boundary component, one of the filled manifolds contains a lens space summand. However, a link complement in S^3 cannot contain a lens space summand. As the surgered manifolds are link complements in S^3, the exterior of J is reducible, and so J is a split link.

Now, we complete the induction using a similar strategy. Suppose that J has n-components. By assumption, all proper sublinks are split unions of unlinks and Hopf links. If J has pairwise linking numbers all zero, we can again apply Proposition 3.1. Therefore, up to reordering of the components, we have at least one two-component sublink $J_1 \cup J_2$ which is a Hopf link. Up to mirroring and reordering J_1 and J_2, the surgery coefficient on J_1 is $1/2$. Trivial surgery on J_2 produces a split link as does $1/2$-surgery, since the resulting link is an $(n-1)$-component link satisfying the same hypotheses of the theorem. Therefore, by applealing again to [CGLS87, Corollary 2.4.7], we get that the complement of J is reducible, so J is split. Since all $(n-1)$-component sublinks are split unions of Hopf links and unknots, and because J itself is split, we now have that J is a split union of Hopf links and unknots. □

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