On the Number of Equivalence Classes of p-Colorings of Symmetric Unions

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Abstract

The symmetric union of a knot, a generalization of the connect sum, has been of interest since its description in the 1950s. We study how the number of equivalence classes of p-colorings, a well-studied knot invariant, is affected by an n-twist symmetric union. We construct a class of knots for which for which the number of equivalence classes of p-colorings of a symmetric union depends on n. We also construct a knot for which the number of equivalence classes of p-colorings of a symmetric union is independent of n.

1 Introduction

In 1957, S. Kinoshita and H. Terasaka introduced the *symmetric union*, a generalization of the connect sum of a knot and its mirror image, that introduces a tangle replacement with n twists across the axis of symmetry. We give a formal definition of symmetric union in Section 2 and for now, we simply denote a symmetric union of a knot K with n twists by $S_n(K)$. There are several knot invariants of symmetric unions that do not depend on the number of twists, n. For instance, when n is even we have that $\det(S_n(K)) = (\det(K))^2$. Another example is that the Alexander polynomial of $S_n(K)$ depends only on the parity of n; that is, for even n, $\Delta_{S_n(K)} = {\Delta_K}^2$ and for odd n, $\Delta_{S_n(K)} = \Delta_K$. [3] A final example is that for a prime p, $S_n(K)$ is p-colorable if and only if K is p-colorable. Throughout the paper, we assume that p is an odd prime and that n is a positive even integer.

In this paper, we seek to study the relationship between more refined knot invariants of K and $S_n(K)$. In particular, we investigate the relationship between the number of equivalence classes p-colorings of a knot K and $S_n(K)$. Clearly, the number of equivalence classes of p-colorings is a more nuanced knot invariant than both p-colorability and knot determinant. As such, one might anticipate a more complicated relationship between the number of equivalence classes of p-colorings of K and $S_n(K)$ that possibly depends on n. On the other hand, the previously mentioned knot invariants do not depend n. We investigate this further in the sections to follow.

In Section 2, we provide background material necessary for our results. In Section 3, we present our main results. Finally, we provide a conclusion and discussion of further areas of research in Section 4.

2 Background

2.1 Symmetric Unions

Suppose K is a knot. Denote the mirror of K by $m(K)$ and denote the connect sum of K and $m(K)$ by $K\#m(K)$. We use the definition of symmetric union provided in [5].

Definition 1. *Let K be a knot with oriented diagram D. A* symmetric union diagram *of D is obtained by replacing an elementary 0-tangle with an elementary n-tangle T_n with $n \neq 0, \infty$ along the axis of mirror symmetry in the symmetric diagram of $K\#m(K)$. A knot which admits a symmetric union diagram is called a* symmetric union.

In an abuse of notation, we denote both a symmetric union of K and its knot diagram by $S_n(K)$ and clarify the difference when necessary. While connected sum is a well defined operation, the symmetric union $S_n(K)$ depends both of the diagram of K and the placement of the tangle region. Let us say that an *interior symmetric union diagram* of an oriented knot K is a symmetric union diagram where the n-tangle is placed in the interior of the region in the diagram which is bounded by the strands joined in the connect sum, as shown in Figure 1. We say that a knot which admits an interior symmetric union diagram is called a *interior symmetric union.* In a similar abuse of notation, we refer to both a interior symmetric union of K and its diagram as simply an interior symmetric union.

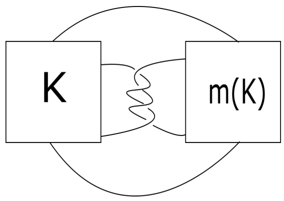


Figure 1: Interior Symmetric Union

2.2 Seifert Surfaces, Graphs, and Matrices

Suppose K is a knot with diagram D. The corresponding *checkerboard surface*, $S \subset \mathbb{R}^2$ is obtained by coloring the pieces of $\mathbb{R}^2 \setminus D$ either white or black such that bordering pieces are colored differently and all black regions are bounded, then joining the black regions by twisted bands passing through the crossing points. Note that the checkerboard surface has the knot K as its boundary but it need not be orientable. If the checkerboard surface it orientable, then it is a *Seifert surface of K*, an orientable surface that has K as its boundary. We define the *Seifert graph* of the checkerboard surface to have vertices corresponding to the black regions of $\mathbb{R}^2 \setminus D$ and edges between vertices whose regions share a crossing. See Figure 2 for an illustration of a checkerboard surface of the trefoil and the corresponding Seifert graph.

It is easy to see that a Seifert graph G is always planar, by construction. In fact, the cycles that bound the interior regions of the planar embedding of G inherited from the checkerboard surface form a basis for $H_1(G)$. We refer to this collection of cycles as the *preferred basis.* We remark that the preferred basis of $H_1(G)$ is determined

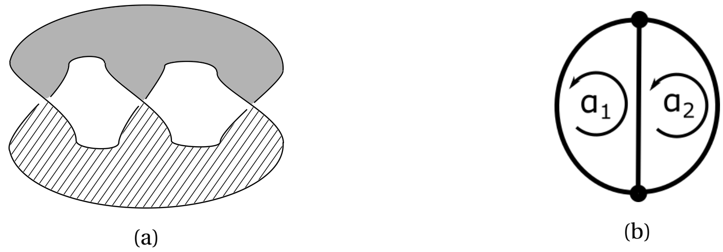


Figure 2: Checkerboard Surface and Seifert Graph of the Trefoil

uniquely by the knot diagram D, up to orientation of its elements. The preferred basis of the Seifert Graph is shown in Figure 2.

Let the closed paths $\alpha_1, \cdots, \alpha_m$ be generators of $H_1(S)$. We consider $S \times [0,1]$ to be a thickened version of the surface S and 'lift' each closed path α_i into two paths by α_i and $\alpha_i^\sharp$, where α_i is a closed path on $S \times \{0\}$ and $\alpha_i^\sharp$ is a closed path on $S \times \{1\}$.

The *Seifert matrix*, V, is a square matrix with m rows and columns whose entries are the linking numbers of all pairs of loops α_i and $\alpha_j^\sharp$. We say that V is the *preferred Seifert matrix* if the generators $\alpha_1, \ldots, \alpha_m$ of $H_1(S)$ correspond to the cycles in the preferred basis for $H_1(G)$. For more information on Seifert matrices, we direct the reader to [4]. Note that $M = V + V^{\mathrm{T}}$ is a presentation matrix for the first homology group of the 2-fold branched covering along K [6]. Hereafter, we shall refer to M as a presentation matrix of the knot K and we say that M is the *preferred presentation matrix* if V is the preferred Seifert matrix.

2.3 Pretzel Knots

For non-zero integers $q_1, q_2, \ldots, q_m$, the corresponding *pretzel link*, is given by the standard projection shown in Figure 3 where the q_i indicates q_i crossing points with sign corresponding to sign(q_i). Such a pretzel link is denoted by $P(q_1, q_2, \ldots, q_m)$. Here, we list a few basic results on pretzel links. First, $P(q_1, q_2, \ldots, q_m)$ is a knot if and only if either both n and all p_i are odd or there is exactly one p_i that is even. Moreover, if $q_1, q_2, \ldots, q_m$ are all odd then the checkerboard surface of the standard projection of $P(q_1, q_2, \ldots, q_m)$ is orientable. Finally, if $q_1, q_2, \ldots q_m$ are all of the same sign then $P(q_1, q_2, \ldots, q_m)$ is alternating. Throughout this paper, we consider only the standard projection of a pretzel knot. As such, notions that depend on a knot diagram, such as interior symmetric union and preferred Seifert matrix, will implicitly use the standard projection of a pretzel knot.

2.4 p-Colorability

Given a knot diagram D and an odd prime p, a p-coloring of D is an assignment of integers modulo p (colors) to the arcs of D that satisfies certain conditions. More precisely, let A be the set of arcs of the diagram D. A *p-coloring* of D is a mapping $\pi : A \to \mathbb{F}_p$ such that for every crossing with over-arc c_i and under-arcs c_j and c_{j+1}, we have that $\pi(c_j) + \pi(c_{j+1}) - 2\pi(c_i) \equiv 0 \mod p$. See Figure 4 for an illustration of a colored crossing as well as a 3−coloring of the trefoil knot.

Note that the constant mapping, $\pi(c) = m$ for all $c \in A$, is a valid coloring for each $1 \leq m \leq p$. These colorings are called the *trivial colorings* of K and are all equivalent.

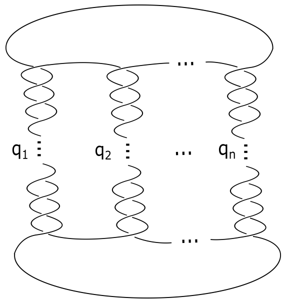
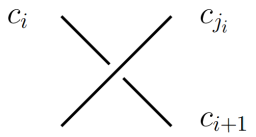
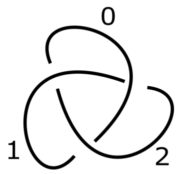


Figure 3: General Pretzel Link $P(q_1, q_2, \ldots, q_m)$



(a) Colored Crossing



(b) 3-Coloring of the Trefoil

Figure 4: Illustrations of p-Colorings.

In general, two p-colorings π_1 and π_2 are said to be equivalent if there is a permutation ϕ of the colors $\{1, \ldots, m\}$ such that $\phi \circ \pi_1 = \pi_2$. It is well-known that the number of equivalent colorings is a knot invariant. We remark that it is not always the case that a permutation of a p-coloring results in another p-coloring. See [1] for details.

To study the number of equivalence classes of colorings of a knot, we use a formula given by Kauffman et al., in [1]. Note that our notions of p-nullity, and thus the result, are phrased slightly differently since we are using Seifert matrices while Kauffman et al. use coloring matrices. The two are equivalent since both the presentation matrix $M = V + V^{\mathrm{T}}$ obtained from the Seifert matrix and the coloring matrix present first homology group of the 2-fold branched covering along a link [2, 3]. The result on the number of equivalence classes of colorings of a knot is given below.

Definition 2. *Let p be an odd prime. Suppose K is a knot with Seifert matrix V and presentation matrix $M = V + V^{\mathrm{T}}$. The p-nullity of K is the nullity of M when the entries are considered to be in $\mathbb{F}_p$.*

Proposition 2.1. *[Proposition 3.1 in [1]] Let p be a prime and n be a positive integer. A knot K with p-nullity n has*

$$\frac{p^n - 1}{p - 1}$$

equivalence classes of p-colorings.

3 Main Results

For notational convenience, denote by $D_k(x)$ the diagonal matrix with zeros in all but the kth diagonal entry, and the value x in that entry instead. Symbolically,

$$(D_k(x))_{ij} = x \cdot \delta_{ik} \cdot \delta_{jk}.$$

Lemma 3.1. *Suppose K is a knot with orientable checkerboard surface S, preferred Seifert matrix V, and preferred presentation matrix $M = V + V^{\mathrm{T}}$. If $S_n(K)$ is an interior symmetric union of K, then $S_n(K)$ has the presentation matrix:*

$$\left(\begin{array}{c|c} M & 0 \\ \hline D_k(n) & M \end{array}\right),$$

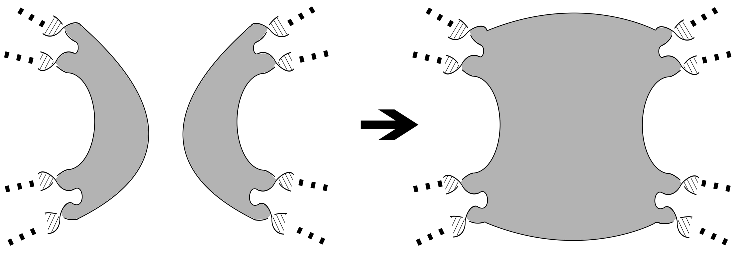
where k is the coordinate of the (unique) preferred-basis element of K affected by the symmetric union.

Proof. The proof has four main steps.

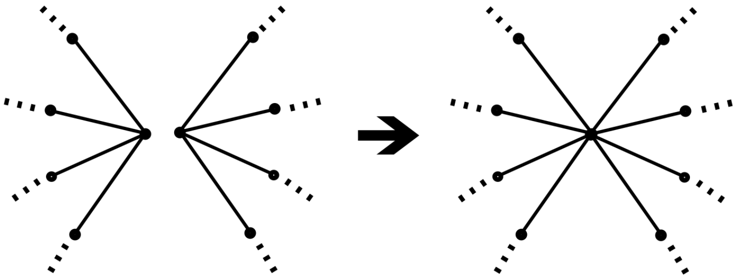
Step 1. Denote the members of the preferred basis of $H_1(S)$ as $\alpha_1,\ldots,\alpha_m$. Notice that the checkerboard surface for the mirror diagram $m(K)$ has an analogous preferred basis, and so denote these basis elements $\alpha'_1,\ldots,\alpha'_m$. Letting $K\#m(K)$ denote the knot diagram obtained by symmetrically connecting the two diagrams without creating any new crossings, one can see that since K has an orientable bounded checkerboard surface, $K\#m(K)$ must have one as well; in particular, it consists of the surfaces of the two components being "joined" together by merging two corresponding checkerboard regions, see Figure 5a. In terms of the Seifert graphs, the Seifert graph of $K\#m(K)$ is obtained by identifying two corresponding vertices of each of the component graphs, as in Figure 5b. Observe that a Seifert matrix for the connect sum of K with its mirrored knot, $K\#m(K)$ is given by:

$$\left(\begin{array}{c|c} V & 0 \\ \hline 0 & -V \end{array}\right),$$

where the preferred basis is simply the union of the two component bases, viz. $(\alpha_1,\ldots,\alpha_m,\alpha'_1,\ldots,\alpha'_m)$. Notice that the negative sign comes from choosing the mirror orientation for each α'_j relative to α_j, and that $\mathrm{lk}\left(\alpha_i,\alpha'^{\sharp}_j\right) = \mathrm{lk}\left(\alpha'_i,\alpha^{\sharp}_j\right) = 0$ for all i, j since such loops are unlinked.



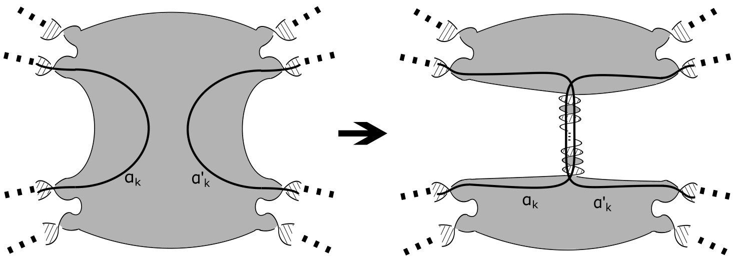
(a) Checkerboard Surface



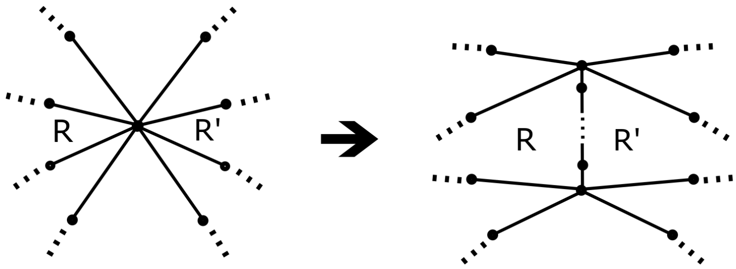
(b) Seifert Graph

Figure 5: Effects of the Connect Sum Process

Step 2. Since an interior symmetric union is being performed, the checkerboard surface for $S_n(K)$ will be the same as the one for $K\#m(K)$ except with $n-1$ new checkerboard regions which "split" the central region. Since n is even, one can extend the orientation of the checkerboard surface of the connect sum to obtain an orientation of the checkerboard surface of this symmetric union. See Figure 6a for an illustration of this orientation. Moreover, the Seifert graph of $S_n(K)$ will be the same as that for the connect sum except that central vertex will be split into a path of $n+1$ vertices. The edges that were connected to the central vertex will be split into two types — those which connect to the "top" vertex and those which connect to the "bottom" vertex. This process will still preserve the mirror symmetry since the tangle region was placed symmetrically, and as a result a pair of interior regions in the graph will have n more edges added to their common boundary, as in Figure 6b. Denote these regions as R and R'.



(a) Checkerboard Surface



(b) Seifert Graph

Figure 6: Effects of an Added n-Tangle

Step 3. A preferred Seifert matrix for $S_n(K)$, denoted $\tilde{V}_n$ is:

$$\tilde{V}_n = \left(\begin{array}{c|c} V & 0 \\ \hline 0 & -V \end{array}\right) + \left(\begin{array}{c|c} D_k\left(\frac{n}{2}\right) & D_k\left(\frac{n}{2}\right) \\ \hline D_k\left(\frac{n}{2}\right) & D_k\left(\frac{n}{2}\right) \end{array}\right).$$

To see this, first observe that, from the results in step 2, a preferred basis for $S_n(K)$ is given by $(\alpha_1,\ldots,\hat{\alpha}_k,\ldots,\alpha_m,\alpha'_1,\ldots,\hat{\alpha}'_k,\ldots,\alpha'_m)$, where k is the coordinate of the unique preferred basis elements which corresponds to the interior regions R and R' (see Figure 6b). Thus, the corresponding basis loops are denoted $\hat{\alpha}_k$ and $\hat{\alpha}'_k$ instead of α_k and α'_k, whereas all other loops are exactly the same as in the connect sum. It follows that the only preferred Seifert matrix elements that will be different are those at positions (k,k), $(k,k+m)$, $(k+m,k)$, and $(k+m,k+m)$. Since n is even and the mirror orientation is given to the loops on the mirror surface, the linking numbers (and hence the respective matrix elements) become:

$$\begin{aligned}
\mathrm{lk}\left(\hat{\alpha}_k,\hat{\alpha}^{\sharp}_k\right) &= \mathrm{lk}\left(\alpha_k,\alpha^{\sharp}_k\right)+\frac{n}{2}\\
\mathrm{lk}\left(\hat{\alpha}_k,\hat{\alpha}'^{\sharp}_k\right) &= \mathrm{lk}\left(\alpha_k,\alpha'^{\sharp}_k\right)+\frac{n}{2}\\
\mathrm{lk}\left(\hat{\alpha}'_k,\hat{\alpha}^{\sharp}_k\right) &= \mathrm{lk}\left(\alpha'_k,\alpha^{\sharp}_k\right)+\frac{n}{2}\\
\mathrm{lk}\left(\hat{\alpha}'_k,\hat{\alpha}'^{\sharp}_k\right) &= \mathrm{lk}\left(\alpha'_k,\alpha'^{\sharp}_k\right)+\frac{n}{2}.
\end{aligned}$$

The desired form of $\tilde{V}_n$ follows.

Step 4. By step 3, a presentation matrix for $S_n(K)$ may be computed as

$$\tilde{V}_n + \tilde{V}_n^{\mathrm{T}} = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & -M \end{array}\right) + \left(\begin{array}{c|c} D_k(n) & D_k(n) \\ \hline D_k(n) & D_k(n) \end{array}\right).$$

Applying elementary row and column operations:

$$\begin{aligned} &\rightsquigarrow \left(\begin{array}{c|c} M & -M \\ \hline 0 & -M \end{array}\right) + \left(\begin{array}{c|c} D_k(n) & 0 \\ \hline D_k(n) & 0 \end{array}\right) \\ &\rightsquigarrow \left(\begin{array}{c|c} M & 0 \\ \hline 0 & -M \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline D_k(n) & 0 \end{array}\right) \\ &\rightsquigarrow \left(\begin{array}{c|c} M & 0 \\ \hline D_k(n) & M \end{array}\right). \end{aligned}$$

■

The nullity of a matrix can be easily discerned from its Smith Normal form. The following lemma relates the Smith Normal form of a typical matrix arising from a symmetric union to the Smith Normal form of the original knot's presentation matrix.

Lemma 3.2. *Let M be an $m \times m$ symmetric matrix such that the kth row can be written as a linear combination of the other columns. Furthermore, suppose M has Smith Normal form given by:*

$$SN(M) = \begin{pmatrix} \alpha_1 & & & \\ & \ddots & & \\ & & \alpha_{m-1} & \\ & & & 0 \end{pmatrix}$$

Then

$$SN\left(\begin{array}{c|c} M & 0 \\ \hline D_k(n) & M \end{array}\right) = \begin{pmatrix} \alpha_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \alpha_{n-1} & & & & & \\ & & & \alpha_1 & & & & \\ & & & & \ddots & & & \\ & & & & & \alpha_{n-1} & & \\ & & & & & & n & \\ & & & & & & & 0 \end{pmatrix}$$

Proof. Since row and column swaps do not affect the Smith Normal Form of a matrix, one may assume without loss of generality that $k = m$. Let M' denote the $m \times (m-1)$ matrix obtained by deleting the last column of M. Then, since the last column of M is a linear combination of the others, and moreover, by the symmetry of M, the last row is a linear combination of the others, by elementary row and column operations:

$$\left(\begin{array}{ccc|c} & M & & \mathbf{0} \\ \hline 0 & \cdots & 0 & \\ \vdots & \ddots & \vdots & M \\ 0 & \cdots & n & \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} & M' & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \mathbf{0} \\ \hline 0 & \cdots & & 0 & M'^{\mathrm{T}} \\ \vdots & \ddots & & \vdots & \\ 0 & \cdots & & n & 0 \cdots 0 \end{array}\right)$$

Now, it is easy to see that all the remaining row and column operations required to take M' and its transpose to their Smith Normal forms will not involve the mth column or the last row. Hence, the above matrix can be reduced to:

$$\left(\begin{array}{cccc|cccc} \alpha_1 & & & 0 & & & & \\ & \ddots & & \vdots & & \mathbf{0} & & \\ & & \alpha_{n-1} & 0 & & & & \\ 0 & \cdots & 0 & 0 & & & & \\ \hline 0 & \cdots & & 0 & \alpha_1 & & & 0 \\ \vdots & \ddots & & \vdots & & \ddots & & \vdots \\ & & & & & & \alpha_{n-1} & 0 \\ 0 & \cdots & & n & 0 & \cdots & \cdots \cdots & 0 \end{array}\right)$$

which clearly can be reduced to the desired form. ■

Lemma 3.3. *Let $P(q_1, q_2, q_3)$ be a pretzel knot with q_i odd for each i. Then the preferred Seifert matrix is*

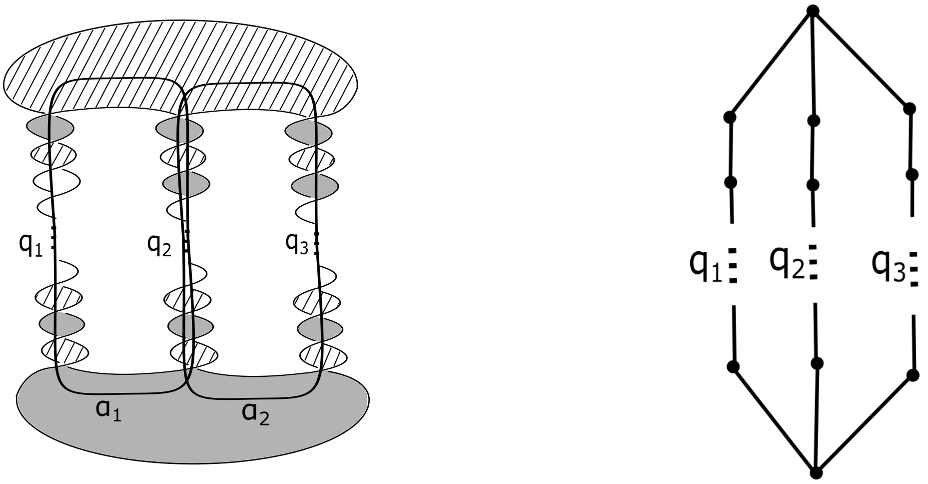
$$V = \begin{pmatrix} \frac{q_1+q_2}{2} & \frac{q_2\pm 1}{2} \\ \frac{q_2\mp 1}{2} & \frac{q_2+q_3}{2} \end{pmatrix}$$

so that the preferred presentation matrix is

$$M = V + V^{\mathrm{T}} = \begin{pmatrix} q_1 + q_2 & q_2 \\ q_2 & q_2 + q_3 \end{pmatrix}$$

Proof. Since each q_i is odd, the checkerboard surface, S, shown in Figure 7a is orientable and hence is a Seifert surface. This surface corresponds to a Seifert graph as in Figure 7b. It is clear from Figure 7a that the preferred basis for $H_1(S)$ is α_1, α_2. Since each q_i is odd, $q_1 + q_2$ and $q_2 + q_3$ are both even, and so the linking numbers between α_1 and $\alpha_1^\sharp$ and between α_2 and $\alpha_2^\sharp$ are $\frac{q_1+q_2}{2}$ and $\frac{q_2+q_3}{2}$ respectively. Notice that so far, these values do not depend on the orientation of the loops. Now, orient the loops so that the strands are pointing in the same direction in the central twisting region. Then, since q_2 is odd, the linking number between α_1 and $\alpha_2^\sharp$ will be $\frac{q_2\pm 1}{2}$, depending on the choice of how the "sharp" loop is pushed off. In the other case, the "plus" will become "minus" and vice versa.

■



(a) Pretzel Knot with Preferred Basis

(b) Seifert Graph

Figure 7: A pretzel knot and Associated Seifert Graph.

Theorem 3.4. *Let K be an alternating pretzel knot with prime determinant p given by $P(q_1, q_2, q_3)$ such that q_i is odd for each $i = 1,2,3$. Then for $n \in 2\mathbb{Z}$, the number of equivalence classes of p-colorings of an interior symmetric union $S_n(K)$ is given by:*

$$\begin{cases} p+1 & p \mid n, \\ 1 & p \nmid n \end{cases}$$

Proof. By Lemma 3.3, a preferred presentation matrix for K is given by:

$$M = \begin{pmatrix} q_1 + q_2 & q_2 \\ q_2 & q_2 + q_3 \end{pmatrix} \tag{1}$$

Observe first that since $\det(K) = p$, M has Smith Normal form:

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pmod{p}$$

Since K is alternating, all the q_is have the same sign. By equation (1),

$$p = \det(M) = q_1 q_2 + q_1 q_3 + q_2 q_3$$

and hence for each i,

$$0 < |q_i| < |p|$$

Moreover,

$$q_3 \begin{pmatrix} q_1 + q_2 \\ q_2 \end{pmatrix} + q_1 \begin{pmatrix} q_2 \\ q_2 + q_3 \end{pmatrix} = \begin{pmatrix} p \\ p \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod{p}$$

Therefore, in $\mathbb{F}_p$ coefficients, since q_1 and q_3 are units, each column of $\bar{M}$ (the matrix reduced mod p) is a linear combination of the other.

By Lemmas 3.1 and 3.2, the Smith Normal form of the presentation matrix for $S_n(K)$ with coefficients in $\mathbb{F}_p$ is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{n} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\bar{n}$ denotes the equivalence class of n in $\mathbb{F}_p$. It is clear that the nullity of this matrix is:

$$\begin{cases} 2 & \text{if } p \mid n \\ 1 & \text{if } p \nmid n \end{cases}$$

The desired result follows from Proposition 2.1. ■

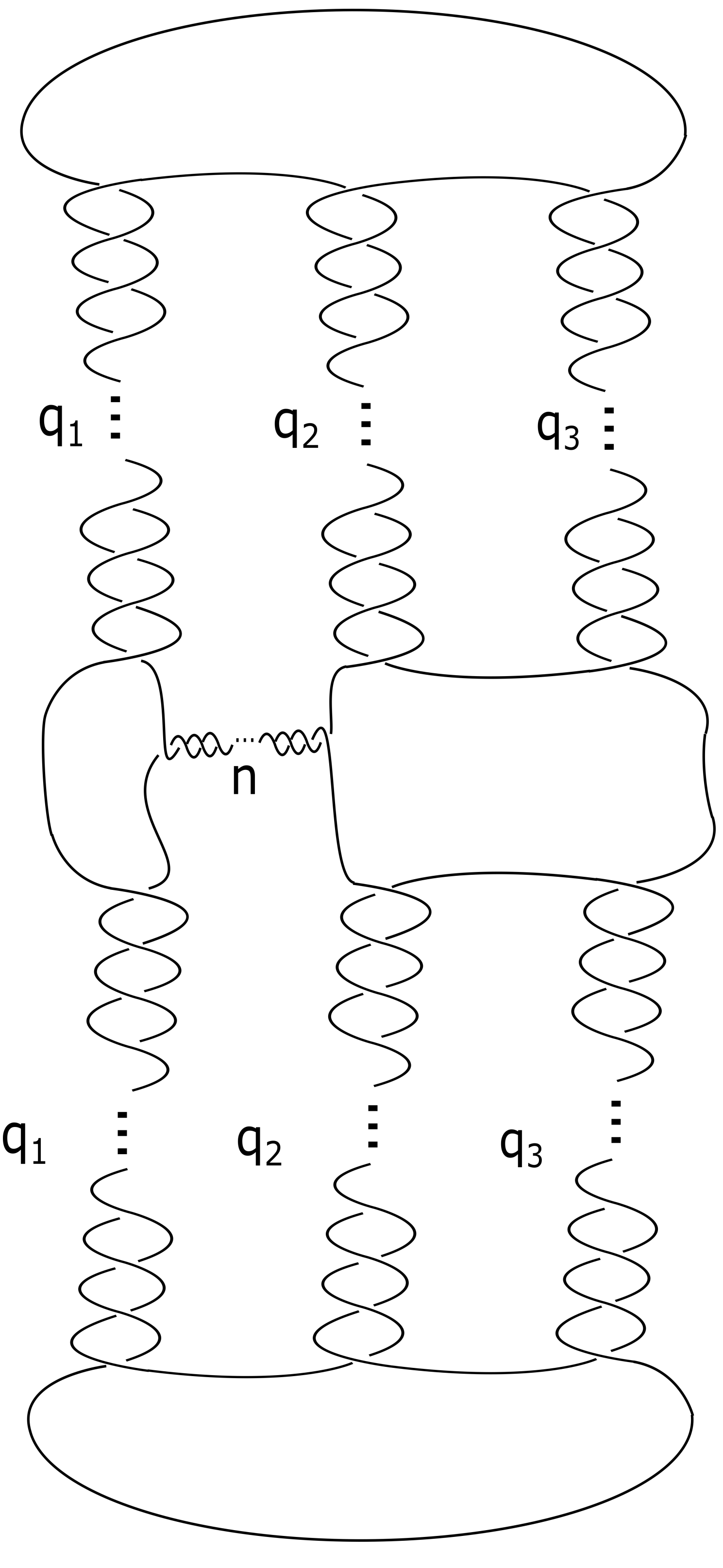


Figure 8: A Generic Interior Symmetric Union of a Pretzel Knot

Theorem 3.5. *There exist symmetric unions of the Pretzel knot $K := P(11, 21, -7)$ such that the number of equivalence classes of 7-colorings do not depend on the number of twists.*

Proof. By Lemma 3.3, K has preferred presentation matrix:

$$\begin{pmatrix} 32 & 21 \\ 21 & 14 \end{pmatrix}$$

Notice that the determinant is 7. We may construct a symmetric union $S_n(K)$ of K where the tangle region is placed as shown in Figure 8. Hence, for n even, by Lemma 3.1, $S_n(K)$ has presentation matrix:

$$\begin{pmatrix} 32 & 21 & 0 & 0 \\ 21 & 14 & 0 & 0 \\ n & 0 & 32 & 21 \\ 0 & 0 & 21 & 14 \end{pmatrix}$$

Reducing to coefficients in $\mathbb{F}_7$ gives:

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{n} & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\bar{n}$ denotes the equivalence class of n in $\mathbb{F}_7$. Thus, it is clear that the above matrix has 7-nullity equal to 2 so that by Proposition 2.1, the number of equivalence classes of 7-colorings is always 8, and is thus independent of n. ■

Remark. It is important to note that Theorem 3.5 has used Lemma 3.1, which assumed that the tangle affected the kth column in the Seifert matrix. If the lemma were modified so that the tangle only affected the second (as opposed to the first) loop, the presentation matrix would have been of the form

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & \bar{n} & 0 & 0 \end{pmatrix} \mod 7,$$

which has a rank that is dependent on $\bar{n}$ just as in Theorem 3.4.

4 Discussion

We have shown in Theorem 3.4 that for the class of 3-tangle alternating pretzel knots with prime determinant p, the number of equivalence classes of p-colorings of their interior symmetric union depends solely on n, the number of twists in the symmetric union. In Theorem 3.5 and its associated remark, we have further shown that this claim does not hold in general for non-alternating pretzel knots with prime determinant. In particular, we have shown that this knot invariant may differ between different symmetric unions of the same knot.

In the future, we would like to investigate the relationship between the number of equivalence classes of p-colorings of a knot and symmetric unions of that knot in more general settings. In Theorem 3.5, we have shown that the conclusion of Theorem 3.4 does not hold for all 3-tangle pretzel knots, even those with prime determinant. We would like to better understand the necessary conditions for which the number of equivalence classes of colorings of a symmetric union will depend on the number of twists. For instance, we would like to consider alternating pretzel knots with more than 3 twisting regions as well as pretzel knots that are non-alternating.

Secondly, we would like to understand how Lemma 3.1 might be extended to knot diagrams with non-orientable checkerboard surfaces. This would involve reformulating the statement in terms of Gordon-Litherland form rather than Seifert form. Considering Gordon-Litherland form might allow for an analogue of Lemma 3.1 for an odd number of twists and possibly arbitrary symmetric unions.

This work was motivated by the question, 'Are all ribbon knots symmetric unions?' A ribbon knot is a knot that is the boundary of a disc whose only intersections are ribbon singularities. In his article, Lamm showed that all symmetric unions are ribbon knots but the converse remains unknown. In his paper, he found symmetric diagrams for all but one of the 21 prime ribbon knots with up to 10 crossings [3]. We began this work hoping to find a knot invariant that could detect whether a knot could be a symmetric union. However, because we constructed classes of knots for which the number of equivalence classes of colorings of symmetric unions does and does not depend on the number of twists in the twisting region, it seems unlikely that these results will help provide an answer to the question.

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